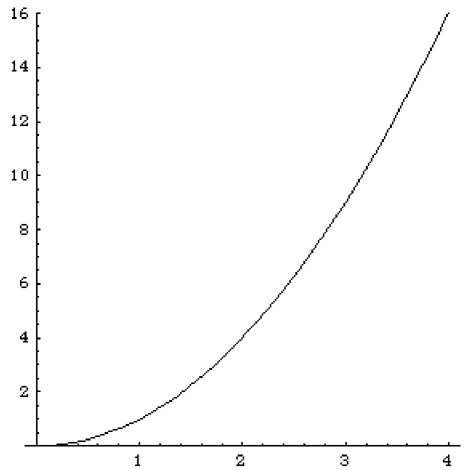
Section 5.1 - Areas and Distances - lecture 2, p. 313 Stewart, 4th Ed.

Recall. Estimating the "area under a curve" using rectangles.
Example. Find the area under $f(x) = x^2$ from $x = 1$ to $x = 4$.



Recall that we used three rectangles and right endpoints to get an (over)estimate of the area of $4 + 9 + 16 = 29$.
(Actual area: 21.)

Question. If we increase the number of rectangles to 6, will we get a better estimate?
Let's compute: width of each rectangle $= \frac{1}{2}$.
Area $\approx$

$$\begin{aligned}
\text{Area} &\approx \underline{\qquad}\cdot\underline{\qquad} + \underline{\qquad}\cdot\underline{\qquad} + \underline{\qquad}\cdot\underline{\qquad} + \underline{\qquad}\cdot\underline{\qquad} \\
&\qquad + \underline{\qquad}\cdot\underline{\qquad} + \underline{\qquad}\cdot\underline{\qquad} \\
&= (\underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad})\cdot\underline{\qquad} \\
&= \underline{\qquad\qquad\qquad}.
\end{aligned}$$

Closer than 29!

Idea. If we keep increasing the number of rectangles, then we will get better and better estimates of the actual area.

n	$\frac{b-a}{n}$	Area
3	1	29
6	$\frac{1}{2}$	
12	$\frac{1}{4}$	≈ 22.906
24	$\frac{1}{8}$	≈ 21.945

Does this **limiting process** sound familiar?

Let's derive a formula for the *exact* area under a curve using limits.
We will need some notation:

Recall. Summation notation: $\sum_{i=1}^{10} 2i$ means $2(1) + 2(2) + 2(3) + \ldots + 2(10)$.

Example. $\sum_{i=1}^{n} (3i-2)^2 =$

Now look at the area under a curve $f(x)$ from $x = a$ to $x = b$. If we have n rectangles (right endpoints), what is the width of each rectangle? ______________

The book calls this Δx.

Call the x-coordinates of the right endpoints $x_1, x_2, \ldots, x_n$. Then the heights of the rectangles are $f(x_1), \ldots, f(x_n)$.

In our previous example, when $n = 6$,
we had $x_1 = 1.5$, $x_2 = 2$, etc.

What are the areas of the rectangles?

$f(x_1)\Delta x, \ldots, f(x_n)\Delta x$.

So with n rectangles, the area estimate is

Picture:

Therefore the area under the curve $f(x)$ from a to b is $\lim_{n\to\infty}$ ____.

Conclusion. Area under a curve =

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $f(x)$ is the "curve" from a to b. We used **right** endpoints to obtain this formula. But we could have also used left endpoints, or midpoints, or any points in the middle of each subinterval.

Motivated by this, the book uses the formula

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where x_i^* is any x-value you want between x_{i-1} and x_i.

Workshop: