Section 6.3 - Volumes by Cylindrical Shells - p. 389 Stewart, 4th Ed.

Recall. the **disk method** was good for finding volumes

- with cross-sectional area $A(x)$ about a *horizontal* axis, or
- with cross-sectional area $A(y)$ about a *vertical* axis.

What about vice-versa?

Example. How would we find the volume of the solid obtained by rotating the region enclosed by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis?

Notice that we've got everything in terms of x's, but we're trying to rotate about a *vertical* axis. This would *not* be fun to do with disks! We'd have to write everything in terms of y and split up the region a bunch of times.

The **Shell Method** is another way to compute volumes, which takes care of those missing cases, namely:

- those with everything in terms of x about a *vertical* axis, or
- those with everything in terms of y about a *horizontal* axis.

Here's how it works: we estimate the volume using cylindrical shells with a certain thickness Δr, and then we let the thickness go to 0 as we make the number of shells infinite.

Example. The volume of a cylinder of radius 4 and height 2 can be gotten by taking 4 shells of height 2 and thickness 1 (see Figure 1).

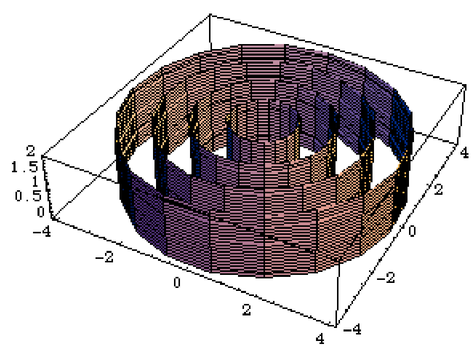

Figure 1: A complicated way to compute the volume of a cylinder

Example. The volume of a cone of radius 4 and height 2 can be estimated using 4 shells of varying heights and thickness 1 (see Figures 2 and 3).

This is an overestimate, of course! But, as always, the more shells we use, the better the estimate.

Volume of a shell with outer radius r_2 and inner radius r_1 is

$$\pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2).$$

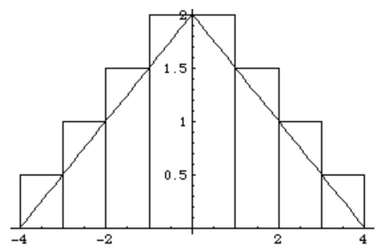

Figure 2: Estimating the volume of a cone — side view of shells

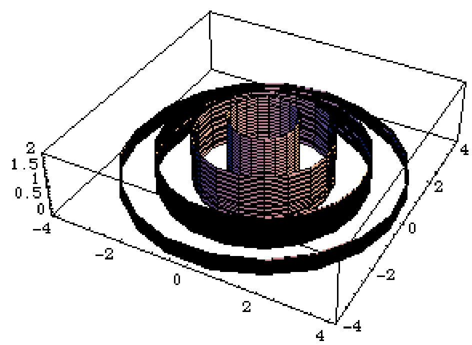

Figure 3: Estimating the volume of a cone — 3D view

Rewrite this in terms of

- $\Delta r = r_2 - r_1$
- *average* of the radii $r = \frac{r_2+r_1}{2}$:

$$\pi h(r_2^2 - r_1^2) = \pi h(r_2 + r_1)(r_2 - r_1)$$

$$= 2\pi h \left(\frac{r_2 + r_1}{2}\right) \Delta r \qquad = 2\pi r h \Delta r = (\text{circumference}) \cdot (\text{height}) \cdot (\text{thickness}).$$

Back to solids of rotation with x's about the y-axis: notice that $r = x$, $\Delta r = \Delta x$, and $h = f(x) - g(x)$. In other words, the volume of a shell with average radius x and height $f(x) - g(x)$ is

$$2\pi x(f(x) - g(x))\Delta x.$$

We add up all these volumes from a to b:

$$\sum_{i=1}^{n} 2\pi x_i(f(x_i) - g(x_i))\Delta x.$$

Then we let $n \to \infty$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i(f(x_i) - g(x_i))\Delta x = \int_a^b 2\pi x(f(x) - g(x))\, dx.$$

Example. Find the volume of the solid formed by rotating the region enclosed by $y = x^2$ and $y = x^3$ about the y-axis.

Solution: The volume is

Example. Same region, but rotate about $x = -2$.

What changes, the height or the radius? ____________. Going back to where we got the formula, we see that the r became an x when the axis of rotation was $x = 0$. Now it's $x + 2$. So the volume becomes

$$\int_0^1 2\pi(\underline{\qquad})(x^2 - x^3)\, dx.$$

Multiply out the integrand to evaluate. Finish here:

We have seen how to find volumes of solids formed by functions of x about a vertical axis. Similarly we can do sideways integration to find volumes of solids formed by functions of y about a horizontal axis:

Example. The region enclosed by $x = y^2$ and $y = x - 2$ about the line $y = 3$.

Solution:

1. The new formula is $\displaystyle\int_c^d 2\pi r h\, dy$. r is now going to be in terms of y, and h is now the difference between two functions of y.

2. Draw a picture:

3. Rewrite the curves in terms of y:

   $x =$ ______________,

   $x =$ ______________.

4. Find intersection points: $c =$ ______, $d =$ ______.

5. The curve ______________ is on the right between -1 and 2. So $h =$ ____________________________.

6. Finally, $r =$ ____________________________, since that's how far away from $y = 3$ we are.

   Therefore the volume is

   (You should get $\frac{45\pi}{2}$.)