Math 75B Practice Midterm I Solutions
Ch. 12, 16, 17 (Ebersole), §§3.10-4.9 (Stewart)

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $f(x)$ is a continuous function and $f(3) = 2$ and $f(5) = -1$, then $f(x)$ has a root between 3 and 5. **[T]** **F**

   This is the Intermediate Value Theorem. Since $f(3) > 0$ and $f(5) < 0$, the function must cross the x-axis somewhere between 3 and 5.

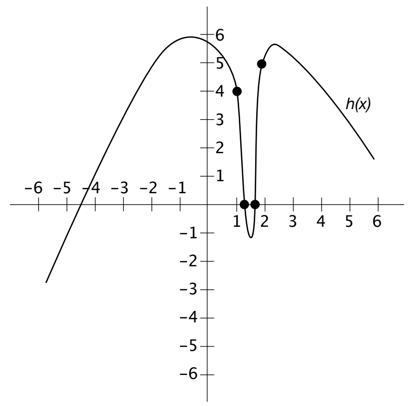
2. The function $g(x) = 2x^3 - 12x + 5$ has 5 real roots. **T** **[F]**

   Since the equation $g'(x) = 6x^2 - 12 = 0$ has 2 solutions, by Rolle's Theorem $g(x)$ has no more than 3 real roots.

3. If $h(x)$ is a continuous function and $h(1) = 4$ and $h(2) = 5$, then $h(x)$ has no roots between 1 and 2. **T** **[F]**

   There was a typo in this problem — the function is called $h(x)$, not $f(x)$. The corrected statement is above.

   But the statement is still false! For example, the function at right is continuous and has $h(1) = 4$ and $h(2) = 5$, but there are **two** roots between 1 and 2.



4. The only x-intercept of $f(x) = x^3 - x^2 + 2x - 2$ is $(1, 0)$. *(Challenge problem!)* **[T]** **F**

   It is true that $(1, 0)$ is an x-intercept of $f(x)$, since $1^3 - 1^2 + 2(1) - 2 = 0$. We can then use Rolle's Theorem to show that $x = 1$ is the *only* root of $f(x)$. We have $f'(x) = 3x^2 - 2x + 2$. If we set $f'(x) = 0$ and use the quadratic formula, we get

$$x = \frac{2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{2 \pm \sqrt{-20}}{6},$$

   which are not real numbers. Therefore there is at most one root of $f(x)$. So $x = 1$ must be it.

Multiple Choice. *Circle the letter of the best answer.*

1. The absolute minimum of $f(x) = -x^2 + 6x + 1$ on the interval $[0, 5]$ is at $x =$

   (a) $\boxed{0}$

   (b) 1

   (c) 2

   (d) 3

   The absolute minimum of a continuous function on a closed interval happens either at a critical number or at one of the endpoints. So we just have to check all of those. We have

$$f'(x) = -2x + 6 \overset{\text{set}}{=} 0$$
$$x = 3$$

   Then $f(0) = 1$, $f(3) = -9 + 18 + 1 = 10$, and $f(5) = -25 + 30 + 1 = 6$. The smallest of these values is 1, and it occurs at $x = 0$.

   By the way, if you happen to notice that $f(x)$ is a parabola opening down, then you don't have to find or check the critical number, since you know it is a local maximum and therefore can't be an absolute minimum.

2. The function $f(x) = \cos x - x$

   (a) is an even function

   (b) is an odd function

   (c) $\boxed{\text{is neither an even nor an odd function}}$

   $f(-x) = \cos(-x) - (-x) = \cos x + x$ (remember that $\cos(-x) = \cos x$!). We have

$$\cos x + x \neq f(x)$$
$$\cos x + x \neq -f(x)$$

   Therefore $f(x)$ is neither even nor odd.

3. The function $f(x) = x^4 - 6x^2$ is increasing on the intervals

   (a) $(0, \sqrt{3})$ only

   (b) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ only

   (c) $(\sqrt{3}, \infty)$ only

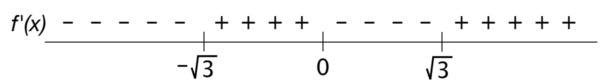
   (d) $\boxed{(-\sqrt{3}, 0) \text{ and } (\sqrt{3}, \infty) \text{ only}}$

$f'(x) = 4x^3 - 12x \overset{\text{set}}{=} 0$. Solving for x, we get

$$4x(x^2 - 3) = 0$$
$$x = 0 \qquad x = \pm\sqrt{3}$$

Now check the number line for $f'(x)$:

$$f'(-2) = (-)(+) = (-), \qquad f'(-1) = (-)(-) = (+), \qquad f'(1) = (+)(-) = (-), \qquad f'(2) = (+)(+) = (+)$$



Therefore the function is increasing on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

4. The function $f(x) = x^4 - 6x^2$ is concave down on the intervals

   (a) $\boxed{(-1, 1) \text{ only}}$

   (b) $(-\sqrt{3}, \sqrt{3})$ only

   (c) $(-\infty, -1)$ and $(1, \infty)$ only

   (d) $(1, \sqrt{3})$ only

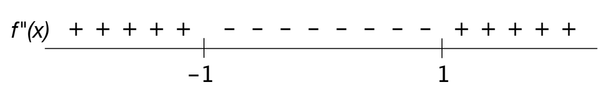
To get concavity we have to get the second derivative. So continuing from the previous question,

$f''(x) = 12x^2 - 12 \overset{\text{set}}{=} 0$. Solving for x, we get

$$12(x^2 - 1) = 0$$
$$x = \pm 1$$

Now check the number line for $f''(x)$:

$$f''(-2) = (+), \qquad f''(0) = (-), \qquad f''(2) = (+)$$



Therefore the function is concave down on the interval $(-1, 1)$.

5. The linear approximation of $f(x) = \sqrt{5-x}$ at $x = 1$ is

   (a) $\boxed{y = -\frac{1}{4}x + \frac{9}{4}}$

   (b) $y = -\frac{3}{4}x + \frac{7}{4}$

   (c) $y = \frac{1}{4}x + \frac{7}{4}$

   (d) $y = -\frac{3}{4}x + \frac{9}{4}$

   The linear approximation of $f(x)$ at $x = 1$ is the same as the equation of the tangent line at $x = 1$.

   $f'(x) = \frac{1}{2}(5-x)^{-1/2}(-1) = -\frac{1}{2\sqrt{5-x}}$, so $f'(1) = -\frac{1}{2\cdot\sqrt{5-1}} = -\frac{1}{4}$. The only answer choice with a slope of $-\frac{1}{4}$ is (5a).

   To make sure that is the right answer, you can also get the y-intercept by plugging in the point $(1, f(1))$ into the equation $y = mx + b$ and solving for b:
   $f(1) = \sqrt{5-1} = 2$, so $2 = -\frac{1}{4}(1) + b$. We get $b = 2 + \frac{1}{4} = \frac{9}{4}$. Therefore the equation is $y = -\frac{1}{4}x + \frac{9}{4}$, as given.

6. If $x_1 = 1$ is a first approximation of a solution to the equation $x^4 = 6 - 3x$, then using Newton's Method the second approximation is $x_2 =$

   (a) $\boxed{\frac{9}{7}}$

   (b) $\frac{5}{7}$

   (c) $\frac{9}{2}$

   (d) $-\frac{5}{2}$

   First we let $f(x) = x^4 + 3x - 6$. Then finding a solution to the equation $x^4 = 6 - 3x$ is the same as finding a root of $f(x)$. The formula for x_2 is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Now $f(1) = 1 + 3 - 6 = -2$ and $f'(x) = 4x^3 + 3$, so $f'(1) = 7$, and we get

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-2}{7} \\ &= 1 + \frac{2}{7} = \boxed{\frac{9}{7}}. \end{aligned}$$

Fill-In.

1. The horizontal asymptote(s) of the function $f(x) = \dfrac{\sqrt{2x^6 - 1}}{x^3 + 2x^2 + 5}$ is/are $\underline{y = \sqrt{2},\ y = -\sqrt{2}}$.

   To find the horizontal asymptotes of a function we take the limits at infinity:

$$\lim_{x\to\infty} \frac{\sqrt{2x^6-1}}{x^3+2x^2+5} = \lim_{x\to\infty} \frac{\sqrt{2x^6-1}}{x^3+2x^2+5} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{2x^6-1}\sqrt{\frac{1}{x^6}}}{(x^3+2x^2+5)\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{\sqrt{(2x^6-1)\left(\frac{1}{x^6}\right)}}{(x^3+2x^2+5)\left(\frac{1}{x^3}\right)}$$
$$= \lim_{x\to\infty} \frac{\sqrt{2-\frac{1}{x^6}}}{1+\frac{2}{x}+\frac{5}{x^3}} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

   The limit as x approaches $-\infty$ is similar, except for the "awful truth" that when $x < 0$, $\frac{1}{x^3} = -\sqrt{\frac{1}{x^6}}$:

$$\lim_{x\to-\infty} \frac{\sqrt{2x^6-1}}{x^3+2x^2+5} = \lim_{x\to-\infty} \frac{\sqrt{2x^6-1}}{x^3+2x^2+5} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$
$$= \lim_{x\to-\infty} \frac{\sqrt{2x^6-1}\left(-\sqrt{\frac{1}{x^6}}\right)}{(x^3+2x^2+5)\left(\frac{1}{x^3}\right)} = \lim_{x\to-\infty} \frac{-\sqrt{(2x^6-1)\left(\frac{1}{x^6}\right)}}{(x^3+2x^2+5)\left(\frac{1}{x^3}\right)}$$
$$= \lim_{x\to-\infty} -\frac{\sqrt{2-\frac{1}{x^6}}}{1+\frac{2}{x}+\frac{5}{x^3}} = -\frac{\sqrt{2}}{1} = -\sqrt{2}.$$

   Therefore the graph of $f(x)$ has a horizontal asymptote at $y = \sqrt{2}$ on the right (as x goes to $+\infty$) and a different horizontal asymptote at $y = -\sqrt{2}$ on the left (as x goes to $-\infty$).

2. Using a tangent line approximation, $\sqrt[3]{126.5} \approx \underline{5.02}$.

   First we get the tangent line approximation to the function $f(x) = \sqrt[3]{x}$ at $x = 125$, since $\sqrt[3]{125} = 5$ is easily computed.

   $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$, so $f'(125) = \frac{1}{3(125)^{2/3}} = \frac{1}{3\cdot 25} = \frac{1}{75}$. This is the slope of the tangent line. The line passes through the point of tangency $(125, 5)$, so we plug these into $y = mx + b$ and solve for b:

$$5 = \frac{1}{75}(125) + b$$
$$5 = \frac{125}{75} + b$$
$$5 = \frac{5}{3} + b$$
$$b = 5 - \frac{5}{3} = \frac{15}{3} - \frac{5}{3} = \frac{10}{3}.$$

So the equation of the tangent line is $y = \frac{1}{75}x + \frac{10}{3}$. Then $\sqrt[3]{126.5}$ is approximated by plugging in $x = 126.5$ to the tangent line:

$$\sqrt[3]{126.5} \approx \frac{1}{75}(126.5) + \frac{10}{3} = \frac{126.5 + 250}{75} = \frac{376.5}{75}.$$

It is okay to leave your answer in that form, but if you are brave you can convert it to a decimal without a calculator: $\frac{376.5}{75} \cdot \frac{4}{4} = \frac{1506}{300} = \frac{502}{100} = \boxed{5.02}$.

3. The absolute maximum value of the function $g(x) = \frac{3}{x-5}$ on the interval $[-3, -1]$ is $\underline{-\frac{3}{8}}$.

   Similar to Multiple Choice #1, the absolute maximum of a continuous function on a closed interval happens either at a critical number or at one of the endpoints. $g(x)$ has a vertical asymptote at $x = 5$, but otherwise is continuous. In particular it is continuous on the interval $[-3, -1]$. We have $g(x) = 3(x-5)^{-1}$, so

$$g'(x) = -3(x-5)^{-2} = \frac{-3}{(x-5)^2} \stackrel{\text{set}}{=} 0$$

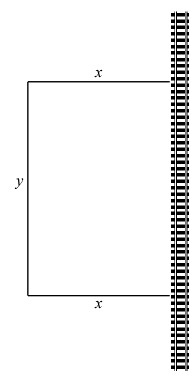
   No solution! Moreover, $g'(x)$ is defined everywhere except $x = 5$, same as $g(x)$. So there are **no critical numbers** of $g(x)$. Therefore the absolute maximum must occur at one of the endpoints of the interval. We have $g(-3) = \frac{3}{-3-5} = -\frac{3}{8}$ and $g(-1) = \frac{3}{-1-5} = -\frac{3}{6} = -\frac{1}{2}$. The larger of these values is $\boxed{-\frac{3}{8}}$.

   Notice! that in Multiple Choice #1 the problem asked for the x-value, whereas this problem asked for the y-value. Be sure to pay careful attention to the wording of problems like this so you know what is being asked for.

4. If a polynomial function $f(x)$ has 3 solutions to the equation $f'(x) = 0$, then $f(x)$ has at most $\underline{4}$ roots.

   According to Rolle's Theorem, if a continuous, differentiable function (such as a polynomial function) has n places where the derivative is 0, then there are at most $n + 1$ real roots.

5. A contractor has 80 ft. of fencing with which to build three sides of a rectangular enclosure. In order to enclose the largest possible area, the dimensions of the enclosure should be $\underline{40\text{ ft.}} \times \underline{20\text{ ft.}}$



   This is a max-min problem. The objective is to **maximize the area**. A formula for the area (see picture) is $A = xy$. To get this formula in terms of a single variable, we need the fact that $2x + y = 80$ (there are only 80 ft. of fencing available). Solving this equation for y, we get $y = 80 - 2x$. So the objective equation becomes

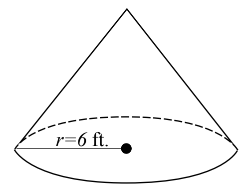
$$A(x) = x(80 - 2x) = 80x - 2x^2.$$

   The domain is $0 \le x \le 40$, but the area is 0 at these endpoints. So the area will be maximized at a critical number between 0 and 40.

$A'(x) = 80 - 4x \overset{\text{set}}{=} 0 \Rightarrow x = 20$. The area is maximized when $x = 20$. This leaves $y = 40$ ft. for the long side ($20 + 20 + 40 = 80$, or equivalently, using the equation $y = 80 - 2x$ we get $y = 80 - 2(20) = 40$).

Work and Answer. *You must show all relevant work to receive full credit.*

1. A cone-shaped roof with base radius $r = 6$ ft. is to be covered with a 0.5-inch layer of tar. Use differentials to estimate the amount of tar required (you may use the formula $V(r) = \frac{2}{9}\pi r^3$ for the volume of the piece of the house covered by the roof).

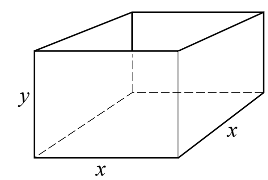


The amount of tar required is approximately $dV = V'(6)dr$, where $dr = \frac{1}{24}$ ft. is the thickness of the tar converted to feet.

[If you need it, on the test I will give you unit conversion formulas such as 12 in. = 1 ft. I will expect you to be able to use this information to calculate conversions such as 0.5 *in.* $= \frac{1}{24}$ *ft. See me* *before the test if you are unsure how to do this.]*

$V'(r) = \frac{2}{3}\pi r^2$, so $V'(6) = \frac{2}{3}\pi \cdot 6^2 = 24\pi$. Therefore $dV = 24\pi \cdot \frac{1}{24} = \boxed{\pi \text{ ft.}^3}$.

2. If 1200 cm^2 of sheet metal is available to make a box with a square base and open top, find the largest possible volume of the box.



The objective of the problem is to **maximize the volume**. A formula for the volume (see picture) is $V = x^2 y$. To get this formula in terms of a single variable, we need the fact that the surface area is $x^2 + 4xy = 1200$. Solving this equation for y, we get

$$4xy = 1200 - x^2$$
$$y = \frac{1200 - x^2}{4x} = 300x^{-1} - \frac{1}{4}x.$$

So the objective equation becomes

$$V(x) = x^2\left(300x^{-1} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3.$$

The domain is $0 \le x \le \sqrt{1200}$, but the volume is 0 at these endpoints. So the volume will be maximized at a critical number between 0 and $\sqrt{1200}$.

$V'(x) = 300 - \frac{3}{4}x^2 \overset{\text{set}}{=} 0 \Rightarrow x^2 = 400 \Rightarrow x = 20$. In other words, the volume is maximized when $x = 20$. Therefore the maximum volume possible is $V = 300(20) - \frac{1}{4} \cdot (20)^3 = 6000 - \frac{8000}{4} = 6000 - 2000 = \boxed{4000 \, \text{cm}^3}$.

Notice! that in Fill-In #5 the problem asked for the dimensions that would maximize the area, whereas this problem asked for the actual maximum volume. Be sure to pay careful attention to the wording of problems like this so you know what is being asked for.

3. Last month I drove to my friend's house 150 miles away. The trip took 3 hours. Explain why there was at least one moment during the trip at which I was driving exactly 49 miles per hour.

   This is the Mean Value Theorem, with a twist.

   The Mean Value Theorem says that at some point in my trip, my instantaneous velocity was the same as the average velocity for the trip, which is $\frac{150-0}{3-0} = 50$ mi./hr. But of course, in order to go 50 mi./hr., I had to go 49 mi./hr. just before that!

4. Estimate the root of $f(x) = x^3 + 2x - 1$ using two iterations of Newton's Method (*i.e.* compute x_3) with the initial guess $x_1 = 0$. *Express your answer as an exact fraction.*

   We use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ twice. $f'(x) = 3x^2 + 2$, so we get

$$x_2 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{2} = \frac{1}{2}. \qquad f(0) = -1 \qquad f'(0) = 2$$

$$x_3 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \frac{1}{2} - \frac{\frac{1}{8}}{\frac{11}{4}} = \frac{1}{2} - \frac{4}{8 \cdot 11} = \frac{1}{2} - \frac{1}{22} = \frac{11-1}{22} = \frac{10}{22} = \boxed{\frac{5}{11}}.$$

$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^3 + 2\left(\tfrac{1}{2}\right) - 1 = \tfrac{1}{8}$$

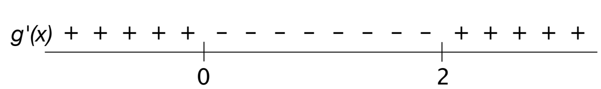
$$f'\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right)^2 + 2 = \tfrac{3}{4} + 2 = \tfrac{11}{4}$$

5. For the function $g(x) = \frac{2}{3}x^3 - 2x^2$,

   (a) find the critical **points** and intervals of increase/decrease

   $g'(x) = 2x^2 - 4x \stackrel{\text{set}}{=} 0$. Solving for x, we get $2x(x-2) = 0 \Rightarrow x = 0, x = 2$. These are the critical **numbers**. $g(0) = 0$ and $g(2) = \frac{2}{3} \cdot 2^3 - 2 \cdot 2^2 = \frac{16}{3} - 8 = \frac{16-24}{3} = -\frac{8}{3}$, so the critical **points** are $(0,0)$ and $\left(2, -\frac{8}{3}\right)$. Now check the number line for $g'(x)$:

$$g'(-1) = (-)(-) = (+), \qquad g'(1) = (+)(-) = (-), \qquad g'(3) = (+)(+) = (+)$$



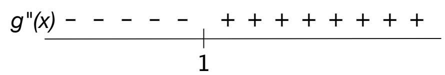
   Therefore the function is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and decreasing on the interval $(0, 2)$.

   (b) find the inflection **points** and intervals of concave up/concave down

   To get concavity we have to get the second derivative. $g''(x) = 4x - 4 = 4(x-1) \stackrel{\text{set}}{=} 0 \Rightarrow x = 1$. So there is an inflection point at $x = 1$. $g(1) = \frac{2}{3} - 2 = \frac{2-6}{3} = -\frac{4}{3}$, so the inflection **point** is $\left(1, -\frac{4}{3}\right)$.

Now check the number line for $g''(x)$:

$$g''(0) = (-), \qquad g''(2) = (+)$$



Therefore the function is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

(c) discuss any symmetry $g(x)$ may or may not have

$g(x)$ has no symmetry, since (in particular) there is an inflection point at $x = 1$ but not at $x = -1$. You can also check that $g(-x) = \frac{2}{3}(-x)^3 - 2(-x)^2 = -\frac{2}{3}x^3 - 2x^2 \neq g(x)$ and $\neq -g(x)$.

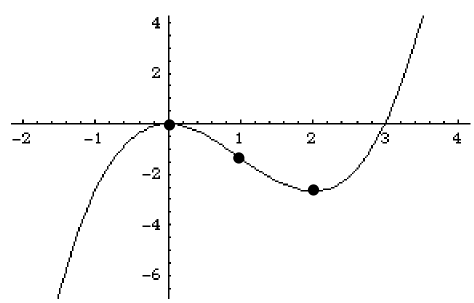
(d) find the equations of any vertical and/or horizontal asymptotes

There are no vertical or horizontal asymptotes because $g(x)$ is a polynomial function.

(e) find the y-intercept

$g(0) = 0$, so the y-intercept is $(0, 0)$.

(f) On the axes at right, sketch an accurate graph of $g(x)$.



6. Prove that the function $f(x) = -x^3 - 6x + 1$ has exactly one real root by completing the following:

(a) Use the Intermediate Value Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at least* one real root.

$f(0) = 1 > 0$ and $f(1) = -1 - 6 + 1 < 0$. Since $f(x)$ is continuous on the interval $[0, 1]$, there must be at least one root between 0 and 1.

(b) Use Rolle's Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at most* one real root.

Suppose $f(x)$ has at least 2 real roots a and b. Since $f(x)$ is a polynomial function, it is continuous and differentiable everywhere. Therefore by Rolle's Theorem there is a number c between a and b so that $f'(c) = 0$. But $f'(x) = -3x^2 - 6 \overset{\text{set}}{=} 0 \Rightarrow x^2 = -2$ has **no solution**. **CONTRADICTION!** Therefore $f(x)$ cannot have more than one real root.