Math 75B Practice Problems for Midterm II – Solutions
Ch. 16, 17, 12 (E), §§4.1-4.5, 2.8 (S)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. The absolute maximum value of $f(x) = \dfrac{1}{x}$ on the interval $[2, 4]$ is 2.   **T** $\boxed{\textbf{F}}$

   If we look at a graph of $f(x) = \dfrac{1}{x}$, we can see that the highest point of the graph on the interval $[2, 4]$ is at $x = 2$. However, the question is asking what the absolute maximum *value* of the function is on this interval which is $f(2) = \dfrac{1}{2}$.

2. If $f(x)$ is a continuous function and $f(3) = 2$ and $f(5) = -1$, then $f(x)$ has $\boxed{\textbf{T}}$ **F**
   a root between 3 and 5.

   This is the Intermediate Value Theorem. Since $f(3) > 0$ and $f(5) < 0$, the function must cross the x-axis somewhere between 3 and 5.

3. The function $g(x) = 2x^3 - 12x + 5$ has 5 real roots.   **T** $\boxed{\textbf{F}}$

   Since the equation $g'(x) = 6x^2 - 12 = 0$ has 2 solutions, by Rolle's Theorem $g(x)$ has no more than 3 real roots.

4. If $h(x)$ is a continuous function and $h(1) = 4$ and $h(2) = 5$, then $h(x)$ has   **T** $\boxed{\textbf{F}}$
   no roots between 1 and 2.

   As an example, the function at right is continuous and has $h(1) = 4$ and $h(2) = 5$, but there are **two** roots between 1 and 2.
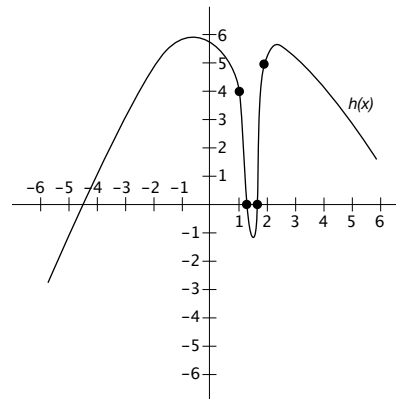
5. The only x-intercept of $f(x) = x^3 - x^2 + 2x - 2$ is $(1, 0)$. *(Challenge problem!)* $\boxed{\textbf{T}}$ **F**

   It is true that $(1, 0)$ is an x-intercept of $f(x)$, since $1^3 - 1^2 + 2(1) - 2 = 0$. We can then use Rolle's Theorem to show that $x = 1$ is the *only* root of $f(x)$. We have $f'(x) = 3x^2 - 2x + 2$. If we set $f'(x) = 0$ and use the quadratic formula, we get

   $$x = \frac{2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{2 \pm \sqrt{-20}}{6},$$

   which are not real numbers. Therefore there is at most one root of $f(x)$. So $x = 1$ must be it.

Multiple Choice. *Circle the letter of the best answer.*

1. The function $f(x) = x^4 - 6x^2$ is increasing on the intervals

   (a) $(0, \sqrt{3})$ only

   (b) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ only
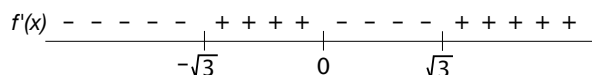
   (c) $(\sqrt{3}, \infty)$ only

   (d) $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ only

   $f'(x) = 4x^3 - 12x \overset{\text{set}}{=} 0$. Solving for x, we get
   $4x(x^2 - 3) = 0$
   $x = 0 \qquad x = \pm\sqrt{3}$
   Now check the number line for $f'(x)$:

   f'(x) − − − − − − + + + + − − − − − + + + + +
         $-\sqrt{3}$ 0 $\sqrt{3}$

   $f'(-2) = (-)(+) = (-), \qquad f'(-1) = (-)(-) = (+),$
   $$f'(1) = (+)(-) = (-), \qquad f'(2) = (+)(+) = (+)$$

   Therefore the function is increasing on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

2. The function $f(x) = \cos x - x$

   (a) is an even function
   (b) is an odd function
   (c) is neither an even nor an odd function

   $f(-x) = \cos(-x) - (-x) = \cos x + x$ (remember that $\cos(-x) = \cos x$!). We have

   $$\cos x + x \neq f(x)$$
   $$\cos x + x \neq -f(x)$$

   Therefore $f(x)$ is neither even nor odd.

3. The function $f(x) = x^4 - 6x^2$ is concave down on the intervals

   (a) $(-1, 1)$ only

   (b) $(-\sqrt{3}, \sqrt{3})$ only

   (c) $(-\infty, -1)$ and $(1, \infty)$ only

   (d) $(1, \sqrt{3})$ only

   To get concavity we have to get the second derivative. So continuing from #1, above,
   $f''(x) = 12x^2 - 12 \overset{\text{set}}{=} 0$. Solving for x, we get

   $$12(x^2 - 1) = 0$$
   $$x = \pm 1$$

   Now check the number line for $f''(x)$:

$$f''(x) \quad + + + + + + \quad - - - - - - - - - \quad + + + + +$$

$$-1 \qquad\qquad 1$$

$$f''(-2) = (+), \qquad f''(0) = (-), \qquad f''(2) = (+)$$

Therefore the function is concave down on the interval $(-1, 1)$.

4. The linear approximation of $f(x) = \sqrt{5 - x}$ at $x = 1$ is

(a) $y = -\dfrac{1}{4}x + \dfrac{9}{4}$

(c) $y = \dfrac{1}{4}x + \dfrac{7}{4}$

(b) $y = -\dfrac{3}{4}x + \dfrac{7}{4}$

(d) $y = -\dfrac{3}{4}x + \dfrac{9}{4}$

The linear approximation of $f(x)$ at $x = 1$ is the same as the equation of the tangent line at $x = 1$.

$f'(x) = \dfrac{1}{2}(5 - x)^{-1/2}(-1) = -\dfrac{1}{2\sqrt{5 - x}}$, so $f'(1) = -\dfrac{1}{2 \cdot \sqrt{5 - 1}} = -\dfrac{1}{4}$. The only answer choice with a slope of $-\dfrac{1}{4}$ is (a).

To make sure that is the right answer, you can also get the y-intercept by plugging in the point $(1, f(1))$ into the equation $y = mx + b$ and solving for b:

$f(1) = \sqrt{5 - 1} = 2$, so $2 = -\dfrac{1}{4}(1) + b$. We get $b = 2 + \dfrac{1}{4} = \dfrac{9}{4}$. Therefore the equation is $y = -\dfrac{1}{4}x + \dfrac{9}{4}$, as given.

5. Suppose $g(x)$ is a polynomial function such that $g(-1) = 4$ and $g(2) = 7$. Then there is a number c between -1 and 2 such that

(a) $g(c) = 1$

(c) $g(c) = 0$

(b) $g'(c) = 1$

(d) $g'(c) = 0$

Since $g(x)$ is a polynomial function it is continuous and differentiable everywhere. Therefore there are two theorems that we might use to answer this question. The first is the Intermediate Value Theorem, which says that between -1 and 2 and any y-value between 4 and 7 there is at least one number c such that $g(c)$ is equal to that y-value. Since none of the answer choices involve y-values between 4 and 7, we go on to the next theorem.

The Mean Value Theorem says that between -1 and 2 there is at least one number c such that $g'(c)$ is equal to the slope of the secant line between the points $(-1, 4)$ and $(2, 7)$, i.e. 1. So (b) is correct.

There is nothing else we can conclude from the information we are given.

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Fill-In.

1. If $f(x)$ is continuous and differentiable on the interval $[2, 4]$ and $f(2) = -1$ and $f(4) = 3$, then there is a number c between 2 and 4 such that *(check **all** that apply)*

   __X__ $f(c) = 0$ __X__ $f(c) = 2$ _____ $f(c) = 4$

   _____ $f'(c) = 0$ __X__ $f'(c) = 2$ _____ $f'(c) = 4$

   Since $f(x)$ is continuous and differentiable on $[2, 4]$, there are two theorems that we might use to answer this question. The first is the Intermediate Value Theorem, which says that between 2 and 4 and any y-value between -1 and 3 there is at least one number c such that $f(c)$ is equal to that y-value. So we check the two answer choices involving y-values between -1 and 3.

   The Mean Value Theorem says that between 2 and 4 there is at least one number c such that $f'(c)$ is equal to the slope of the secant line between the points $(2, -1)$ and $(4, 3)$, i.e. 2. So we check $f'(c) = 2$.

   There is nothing else we can conclude from the information we are given.

2. If a polynomial function $f(x)$ has 3 solutions to the equation $f'(x) = 0$, then $f(x)$ has at most $\underline{4}$ real roots.

   According to Rolle's Theorem, if a continuous, differentiable function (such as a polynomial function) has n places where the derivative is 0, then there are at most $n + 1$ real roots.

3. A contractor has 80 ft. of fencing with which to build three sides of a rectangular enclosure. In order to enclose the largest possible area, the dimensions of the enclosure should be $\underline{40\,\text{ft.}} \times \underline{20\,\text{ft.}}$
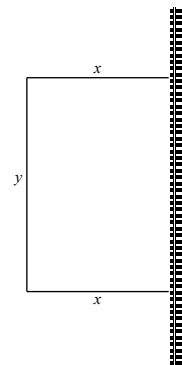
   This is a max-min problem. The objective is to **maximize the area**. A formula for the area (see picture) is $A = xy$. To get this formula in terms of a single variable, we need the fact that $2x + y = 80$ (there are only 80 ft. of fencing available). Solving this equation for y, we get $y = 80 - 2x$. So the objective equation becomes

   $$A(x) = x(80 - 2x) = 80x - 2x^2.$$

   The domain is $0 \le x \le 40$, but the area is 0 at these endpoints. So the area will be maximized at a critical number between 0 and 40.

   $A'(x) = 80 - 4x \overset{\text{set}}{=} 0 \Rightarrow x = 20$. The area is maximized when $x = 20$. This leaves $y = 40$ ft. for the long side ($20 + 20 + 40 = 80$, or equivalently, using the equation $y = 80 - 2x$ we get $y = 80 - 2(20) = 40$).

4. Using a tangent line approximation, $\sqrt[3]{126.5} \approx \underline{5.02}$.

   We can do this problem in two ways: by finding the equation of the tangent line to the function $f(x) = \sqrt[3]{x}$ at $x = 125$ and then plugging in 126.5, or by using differentials.

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Solution 1. $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$, so $f'(125) = \frac{1}{3(125)^{2/3}} = \frac{1}{3 \cdot 25} = \frac{1}{75}$. This is the slope of the tangent line. The line passes through the point of tangency $(125, 5)$, so we plug these into $y = mx + b$ and solve for b:

$$5 = \frac{1}{75}(125) + b$$
$$5 = \frac{125}{75} + b$$
$$5 = \frac{5}{3} + b$$
$$b = 5 - \frac{5}{3} = \frac{15}{3} - \frac{5}{3} = \frac{10}{3}.$$

So the equation of the tangent line is $y = \frac{1}{75}x + \frac{10}{3}$. Then $\sqrt[3]{126.5}$ is approximated by plugging in $x = 126.5$ to the tangent line:

$$\sqrt[3]{126.5} \approx \frac{1}{75}(126.5) + \frac{10}{3} = \frac{126.5 + 250}{75} = \frac{376.5}{75}.$$

It is okay to leave your answer in that form, but if you are brave you can convert it to a decimal without a calculator: $\frac{376.5}{75} \cdot \frac{4}{4} = \frac{1506}{300} = \frac{502}{100} = \boxed{5.02}$.

Solution 2. We have $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$, so

$$dy = \frac{1}{3x^{2/3}} \, dx$$

Now we plug in $x_{\text{easy}} = 125$ and $dx = x_{\text{hard}} - x_{\text{easy}} = 126.5 - 125 = 1.5$, and we obtain

$$dy = \frac{1}{3 \cdot 125^{2/3}} \cdot (1.5) = \frac{1}{3 \cdot 25} \cdot \frac{3}{2} = \frac{1}{50}.$$

Now $y_{\text{approx}} = y_{\text{easy}} + dy = 5 + \frac{1}{50} = \boxed{5.02}$

Graphs.

1. For the function $h(x) = \dfrac{x+5}{x^2 - 9}$,

   (a) Find the equations of any vertical and/or horizontal asymptotes

   $h(x) = \dfrac{x+5}{x^2 - 9} = \dfrac{x+5}{(x+3)(x-3)}$, so there are vertical asymptotes at $\boxed{x = 3 \text{ and } x = -3}$

   The domain of $h(x)$ is $x \neq \pm 3$.

   $\lim\limits_{x \to \pm\infty} \dfrac{x+5}{x^2-9} = 0$ since the denominator has larger degree than the numerator. So there is a horizontal asymptote at $\boxed{y = 0}$

   (b) Find the y-intercept

   $h(0) = \dfrac{5}{-9} = -\dfrac{5}{9}$, so the y-intercept is the point $\boxed{\left(0, -\dfrac{5}{9}\right)}$

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(c) Find the x-intercept

$\dfrac{x+5}{x^2-9} \overset{\text{set}}{=} 0$ gives $x = -5$ as the only solution. So the x-intercept is the point $\boxed{(-5,0)}$

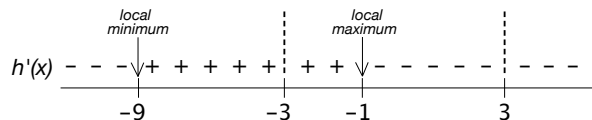(d) Find the critical **points** and intervals of increase/decrease

We have

$$
\begin{aligned}
h'(x) &= \frac{x^2 - 9 - (x+5)\cdot 2x}{(x^2-9)^2} = \frac{x^2 - 9 - 2x^2 - 10x}{(x^2-9)^2} \\
&= \frac{-x^2 - 10x - 9}{(x^2-9)^2} = -\frac{x^2 + 10x + 9}{(x^2-9)^2} \\
&= -\frac{(x+9)(x+1)}{(x^2-9)^2} \overset{\text{set}}{=} 0 \\
&\Rightarrow \ x = -9, \ x = -1.
\end{aligned}
$$

So the critical numbers are $x = -9$, $x = -1$ ($x = \pm 3$ make $h'(x)$ undefined, but they are not in the domain of $h(x)$ to begin with). Now $h(-9) = -\dfrac{1}{18}$ and $h(-1) = -\dfrac{1}{2}$ (check), so the critical points are

Therefore the critical points are $\boxed{\left(-9, -\dfrac{1}{18}\right) \text{ and } \left(-1, -\dfrac{1}{2}\right)}$
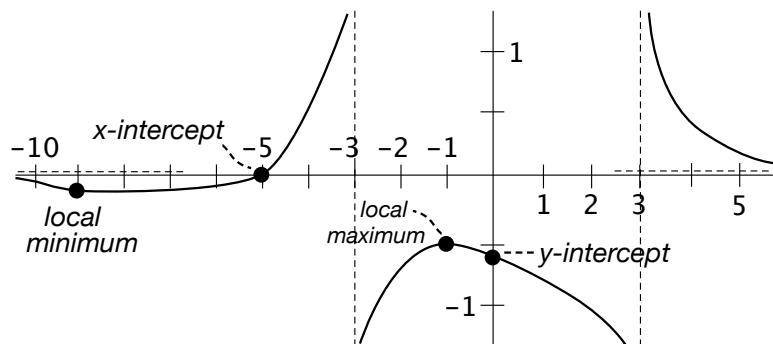
To find the intervals of increase and decrease, we set up a number line for $h'(x)$:

$$
h'(-10) = -\frac{(-)(-)}{(+)} = (-), \qquad h'(-4) = -\frac{(+)(-)}{(+)} = (+),
$$

$$
h'(-2) = -\frac{(+)(-)}{(+)} = (+), \qquad h'(0) = -\frac{(+)(+)}{(+)} = (-), \qquad h'(4) = -\frac{(+)(+)}{(+)} = (-)
$$

Therefore the function is increasing on the intervals $(-9, -3)$ and $(-3, -1)$, and is decreasing on the intervals $(-\infty, -9)$, $(-1, 3)$ and $(3, \infty)$.

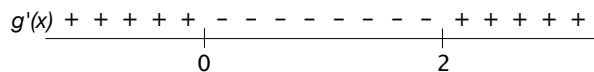(e) On the axes below, sketch an accurate graph of $h(x)$.

2. For the function $g(x) = \frac{2}{3}x^3 - 2x^2$,

(a) find the critical **points** and intervals of increase/decrease

$g'(x) = 2x^2 - 4x \overset{\text{set}}{=} 0$. Solving for x, we get $2x(x-2) = 0 \Rightarrow x = 0$, $x = 2$. These are the critical **numbers**. $g(0) = 0$ and $g(2) = \frac{2}{3} \cdot 2^3 - 2 \cdot 2^2 = \frac{16}{3} - 8 = \frac{16 - 24}{3} = -\frac{8}{3}$, so the critical **points** are $(0,0)$ and $\left(2, -\frac{8}{3}\right)$. Now check the number line for $g'(x)$:

$$g'(-1) = (-)(-) = (+), \qquad g'(1) = (+)(-) = (-), \qquad g'(3) = (+)(+) = (+)$$

g'(x) + + + + + - - - - - - - - - + + + + +
       ————————————|————————————————|——————————————
   0   2

Therefore the function is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and decreasing on the interval $(0, 2)$.

(b) find the inflection **points** and intervals of concave up/concave down

To get concavity we have to get the second derivative. $g''(x) = 4x - 4 = 4(x-1) \overset{\text{set}}{=} 0 \Rightarrow$ $x = 1$. So there is an inflection point at $x = 1$. $g(1) = \frac{2}{3} - 2 = \frac{2 - 6}{3} = -\frac{4}{3}$, so the inflection **point** is $\left(1, -\frac{4}{3}\right)$.

Now check the number line for $g''(x)$: $g''(0) = (-)$, $\qquad g''(2) = (+)$, so we have

g"(x) - - - - - - + + + + + + + + +
       ————————————|————————————————————
                   1
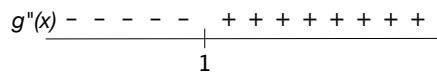
Therefore the function is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

(c) discuss any symmetry $g(x)$ may or may not have

$g(x)$ has no symmetry, since (in particular) there is an inflection point at $x = 1$ but not at $x = -1$. You can also check that $g(-x) = \frac{2}{3}(-x)^3 - 2(-x)^2 = -\frac{2}{3}x^3 - 2x^2 \neq g(x)$ and $\neq -g(x)$.
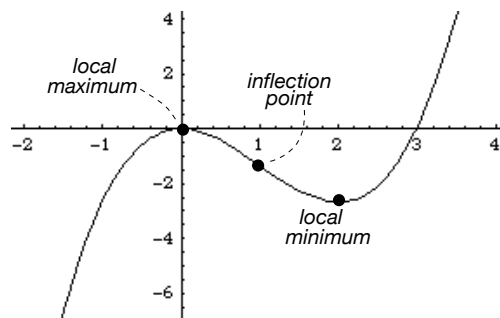
(d) find the equations of any vertical and/or horizontal asymptotes

There are no vertical or horizontal asymptotes because $g(x)$ is a polynomial function.

(e) find the y-intercept

$g(0) = 0$, so the y-intercept is $(0,0)$.

(f) On the axes at right, sketch an accurate graph of $g(x)$.

3. On the axes at right, sketch a graph of a function $f(x)$ satisfying all of the following properties:

- $f(x)$ is an even function

- $(3, 0)$ is a critical point of $f(x)$

- $(4, -1)$ is an inflection point of $f(x)$

- $f(x)$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = -2$

- $f(x)$ is increasing on the intervals $(-\infty, -3)$ and $(0, 3)$

- $f(x)$ is concave up on the intervals $(-\infty, -4)$ and $(4, \infty)$
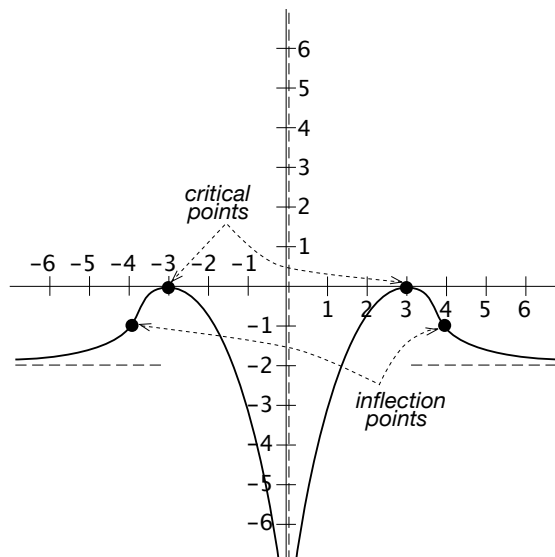
Notice that since $f(x)$ is an even function, it is symmetric about the y-axis. Therefore, since there is a critical point at $(3, 0)$, there must also be a critical point at $(-3, 0)$. Similarly, since there is an inflection point at $(4, -1)$, there must also be an inflection point at $(-4, -1)$. The restrictions above leave very little room for variety in this graph.

4. The graph of a function $f(x)$ is shown, along with the linear approximation at a point (x, y). Fill in each blank with the correct label from the following list: dx, y, dy, and Δy.

The blanks are filled in at right. Notice that Δy is the *actual* change in the y-value measured on the actual graph of the function — in other words, $\Delta y = f(x + \Delta x) - f(x)$ — whereas dy is the *approximate* change measured on the tangent line.
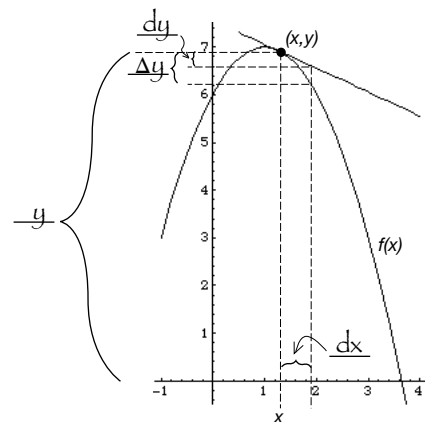
Work and Answer. *You must show all relevant work to receive full credit.*

1. Find the absolute maximum and the absolute minimum values of the function $f(x) = \dfrac{\ln x}{x}$ on the interval $[1, e^2]$.

By the Extreme Value Theorem, the extreme values of the function will be attained either at the endpoints of the interval or at a critical number in the interval. So we must find the critical

numbers. We have

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} \overset{\text{set}}{=} 0$$

$$1 - \ln x = 0$$

$$\ln x = 1$$

$$x = e.$$

So $x = e$ is the only critical number, and it is in the interval. We plug in the critical number and the endpoints to see which y-value is largest and which is smallest. We have

$$f(1) = \frac{\ln(1)}{1} = 0$$

$$f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$$

$$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2}$$

Clearly 0 is the smallest of these y-values, so the absolute minimum value of $f(x)$ on the interval is $\boxed{0}$.

Now $\frac{1}{e} > \frac{1}{3}$ since $e \approx 2.7 < 3$, whereas $\frac{2}{e^2} \approx \frac{2}{7} < \frac{1}{3}$. Therefore the largest number is $\boxed{\frac{1}{e}}$.

2. If 1200 cm^2 of sheet metal is available to make a box with a square base and open top, find the largest possible volume of the box.

This is similar to a question on a recent quiz. Now we get to do the problem all the way through!

The objective of the problem is to **maximize the volume**. A formula for the volume (see picture) is $V = x^2 y$. To get this formula in terms of a single variable, we need the fact that the surface area is $x^2 + 4xy = 1200$. Solving this equation for y, we get

$$4xy = 1200 - x^2$$

$$y = \frac{1200 - x^2}{4x} = 300x^{-1} - \frac{1}{4}x.$$

So the objective equation becomes

$$V(x) = x^2 \left(300x^{-1} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3.$$
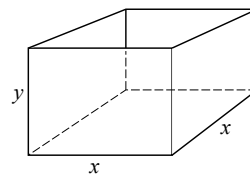
This was exactly the formula I gave you on blue quiz #7 (the green quiz was a little different since the amount of sheet metal was given to be 2400 cm^2).

The domain is $0 \le x \le \sqrt{1200}$, but the volume is 0 at these endpoints. So the volume will be maximized at a critical number between 0 and $\sqrt{1200}$.

$V'(x) = 300 - \frac{3}{4}x^2 \overset{\text{set}}{=} 0 \Rightarrow x^2 = 400 \Rightarrow x = 20$. In other words, the volume is maximized when $x = 20$. Therefore the maximum volume possible is $V = 300(20) - \frac{1}{4} \cdot (20)^3 = 6000 - \frac{8000}{4} = 6000 - 2000 = \boxed{4000 \,\text{cm}^3}$.

Notice! that on the quiz the problem asked for the <u>base length</u> that would maximize the volume, whereas this problem asked for the <u><u>actual maximum volume</u></u>. Be sure to pay careful attention to the wording of problems like this so you know what is being asked for.

3. A cone-shaped roof with base radius $r = 6$ ft. is to be covered with a 0.5-inch layer of tar. Use differentials to estimate the amount of tar required (you may use the formula $V(r) = \dfrac{2}{9}\pi r^3$ for the volume of the piece of the house covered by the roof).

The amount of tar required is approximately $dV = V'(6)dr$, where $dr = \dfrac{1}{24}$ ft. is the thickness of the tar converted to feet.

[If you need it, on the test I will give you unit conversion formulas such as 12 in. = 1 ft. I will expect you to be able to use this information to calculate conversions such as 0.5 in. $= \dfrac{1}{24}$ ft. See me <u>before the test</u> if you are unsure how to do this.]
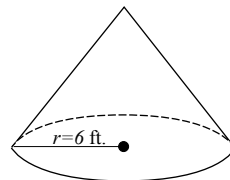
$V'(r) = \dfrac{2}{3}\pi r^2$, so $V'(6) = \dfrac{2}{3}\pi \cdot 6^2 = 24\pi$. Therefore $dV = 24\pi \cdot \dfrac{1}{24} = \boxed{\pi \text{ ft.}^3}$.

4. Prove that the function $f(x) = -x^3 - 6x + 1$ has exactly one real root by completing the following:

(a) Use the Intermediate Value Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at least* one real root.

$f(0) = 1 > 0$ and $f(1) = -1 - 6 + 1 < 0$. Since $f(x)$ is continuous on the interval $[0, 1]$, there must be at least one root between 0 and 1.

(b) Use Rolle's Theorem to show that the function $f(x) = -x^3 - 6x + 1$ has *at most* one real root.

Since $f(x)$ is a polynomial function, it is continuous and differentiable everywhere. Therefore Rolle's Theorem applies. We have $f'(x) = -3x^2 - 6 \overset{\text{set}}{=} 0 \Rightarrow x^2 = -2$ has **no solution**. Therefore $f(x)$ cannot have more than **one** real root.

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