Math 75A Practice Final – Solutions

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. (55 points) *Circle the letter of the best answer.*

1. The vertical asymptotes for the graph of the function $f(x) = \dfrac{x^2+4x+4}{x^2+x-2}$ are

(a) $x = -2$ and $x = 1$

(b) $x = -2$ only

(c) $x = -1$ and $x = 2$

(d) $\boxed{x = 1 \text{ only}}$

(e) $y = 1$ only

A vertical asymptote happens at x-values for which the denominator is equal to zero but the numerator is not. In other words, if you cannot cancel the factor in the denominator that is zero for that value of x. Here $f(x) = \dfrac{(x+2)^2}{(x+2)(x-1)} = \dfrac{x+2}{x-1}$ (as long as $x \neq -2$), so there is a hole at $x = -2$ and a vertical asymptote at $x = 1$ since the $(x-1)$ does not cancel.

2. If $e^{2t} = 5$, then $t =$

(a) $\dfrac{e^2}{5}$

(b) $\boxed{\dfrac{\ln 5}{2}}$

(c) $\ln \dfrac{5}{2}$

(d) $\dfrac{5}{e^2}$

(e) $\dfrac{2}{\ln 5}$

We have $\ln(e^{2t}) = \ln(5)$, so $2t = \ln(5)$ and therefore $\boxed{t = \dfrac{\ln(5)}{2}}$

3. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.

(a) 0

(b) $\boxed{\dfrac{1}{4}}$

(c) $\dfrac{1}{2}$

(d) 1

(e) does not exist

We have

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\\
&= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)}\\
&= \lim_{x\to 0}\frac{x}{x(\sqrt{4+x}+2)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}
\end{aligned}$$

4. Let $g(t) = t^3\cos t$. Find $g'(t)$.

(a) $\boxed{3t^2\cos t - t^3\sin t}$

(b) $-3t^2\sin t$

(c) $3t^2\sin t$

(d) $-3t^2\sin t + t^3\cos t$

(e) $3t^2\cos t + t^3\sin t$

This is a product, so we use the product rule. We have $g'(t) = t^3(-\sin t) + \cos t\cdot 3t^2 = \boxed{3t^2\cos t - t^3\sin t}$

5. At $x = 3$ the graph of the function $f(x) = \dfrac{2x-6}{x-3}$

(a) $\boxed{\text{has a hole}}$

(b) has a vertical asymptote

(c) is at the point $(3, 2)$

(d) has a corner

(e) has a vertical tangent line

We have $f(x) = \dfrac{2x-6}{x-3} = \dfrac{2(x-3)}{x-3}$. Since we can cancel the $(x-3)$'s (given $x \neq 3$), there is a hole in the graph at $x = 3$.

6. The inverse function of $f(x) = 5x^3 - 4$ is $f^{-1}(x) =$

(a) $\dfrac{\sqrt[3]{x}+4}{5}$

(b) $\dfrac{5}{\sqrt[3]{x-4}}$

(c) $\boxed{\sqrt[3]{\dfrac{x+4}{5}}}$

(d) $\dfrac{\sqrt[3]{x+4}}{5}$

(e) $f(x)$ does not have an inverse function

We have

$$
\begin{aligned}
y &= 5x^3 - 4 \\
y + 4 &= 5x^3 \\
\frac{y+4}{5} &= x^3 \\
x &= \sqrt[3]{\frac{y+4}{5}}.
\end{aligned}
$$

Switching x and y, we get $\boxed{f^{-1}(x) = \sqrt[3]{\frac{x+4}{5}}}$

7. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the derivative of the quotient function $\left(\frac{f}{g}\right)(x)$ at $x = 3$, that is, $\left(\frac{f}{g}\right)'(3)$.

   (a) $\boxed{-8}$

   (b) -3

   (c) $-\frac{6}{5}$

   (d) 2

   (e) 8

   Using the quotient rule, we have $\left(\frac{f}{g}\right)'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} = \frac{2 \cdot (-6) - 4 \cdot 5}{2^2} = \frac{-12 - 20}{4} = \boxed{-8}$

8. Evaluate $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6}$.

   (a) 0

   (b) 1

   (c) $\frac{2}{3}$

   (d) $\boxed{\frac{4}{5}}$

   (e) does not exist

   "Dr. Kelm's limit law #1" does not apply, since if you try to plug in $x = 2$ to the function $f(x) = \frac{x^2 - 4}{x^2 + x - 6}$ you get 0 in the denominator. So remember "limit law #2" which is, **Don't give up!** We have

$$
\begin{aligned}
\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6} &= \lim_{x \to 2} \frac{(x+2)(x-2)}{(x+3)(x-2)} \\
&= \lim_{x \to 2} \frac{x+2}{x+3} = \boxed{\frac{4}{5}}
\end{aligned}
$$

9. If $f(x) = \sin(x^2)$, then $f'(x) =$

(a) $\cos(2x)$

(b) $-2x\cos(x^2)$

(c) $\sin(2x) + \cos(x^2)$

(d) $\cos(x^2)$

(e) $\boxed{2x\cos(x^2)}$

Notice that this function is *not* a product, since the x^2 is the input for the sine function ("sine **of** x^2"). So we use the chain rule, and we get $f'(x) = \cos(x^2) \cdot 2x = \boxed{2x\cos(x^2)}$

10. The horizontal asymptote(s) of the function $f(x) = \dfrac{-7x^3 + 5x - 1}{\sqrt{9x^6 + 2}}$ is/are

(a) $y = 0$ only

(b) $y = \dfrac{7}{3}$ only

(c) $\boxed{y = \dfrac{7}{3} \text{ and } y = -\dfrac{7}{3}}$

(d) $y = -\dfrac{7}{3}$ only

(e) $f(x)$ has no horizontal asymptotes

To get horizontal asymptotes we take the limits at $\pm\infty$ of the function. First, approaching $+\infty$ we have

$$\lim_{x\to\infty} \frac{-7x^3 + 5x - 1}{\sqrt{9x^6 + 2}} = \lim_{x\to\infty} \frac{-7x^3 + 5x - 1}{\sqrt{9x^6 + 2}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

(recall that the denominator has the "strength" of x^3 since there is an x^6 under a square root)

$$= \lim_{x\to\infty} \frac{-7 + \frac{5}{x^2} - \frac{1}{x^3}}{\sqrt{9x^6 + 2}\sqrt{\frac{1}{x^6}}}$$

(since $\dfrac{1}{x^3} = \sqrt{\dfrac{1}{x^6}}$)

$$= \lim_{x\to\infty} \frac{-7 + \frac{5}{x^2} - \frac{1}{x^3}}{\sqrt{(9x^6 + 2)\frac{1}{x^6}}}$$

$$= \lim_{x\to\infty} \frac{-7 + \frac{5}{x^2} - \frac{1}{x^3}}{\sqrt{9 + \frac{2}{x^6}}}$$

$$= \frac{-7}{\sqrt{9}} = -\frac{7}{3}$$

(since all other terms go to zero in the limit). Therefore there is a horizontal asymptote (out to the right) at $y = -\dfrac{7}{3}$.

For $-\infty$ we have the "awful truth" that when $x < 0$, $\frac{1}{x^3} = -\sqrt{\frac{1}{x^6}}$, so the limit will come out the opposite of the one above, i.e.

$$\lim_{x\to-\infty} \frac{-7x^3+5x-1}{\sqrt{9x^6+2}} = \frac{7}{3}.$$

Therefore there is a different horizontal asymptote (out to the left) at $y = \frac{7}{3}$.

11. If $f(x) = \sin(x)$, then $f'(x) =$

(a) $\frac{\sin(x+h)-\sin(x)}{h}$

(b) $\frac{\sin(x)+h-\sin(x)}{h}$

(c) $\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$

(d) $\lim_{x\to 0} \frac{\sin(x+h)-\sin(x)}{h}$

(e) $\boxed{\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}}$

The derivative of a function $f(x)$ is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Following this pattern we get the derivative of $f(x) = \sin x$ as above.

Fill-In. (35 points)

1. The domain of the function $f(x) = \sqrt{3x-2}$ is $\underline{\ x \geq \frac{2}{3}\ }$

   We must have $3x - 2 \geq 0$ since it is under a square root. Solving for x we get $x \geq \frac{2}{3}$.

2. The range of the function $g(t) = 5\cos t$ is $\underline{\ -5 \leq y \leq 5\ }$

   $\cos t$ must lie between -1 and 1, so $5\cos t$ must lie between -5 and 5.

3. $\lim_{x\to 4^-} \frac{2}{x-4} = \underline{\ -\infty\ }$

   We know there is a vertical asymptote at $x = 4$, so as we approach 4 from the left, we know the answer will be either ∞ or $-\infty$. If we plug in a number a little less than 4, we get $\frac{2}{\text{(negative)}} = \text{(negative)}$, so the answer must be $-\infty$.

4. The table below *[omitted in solutions]* gives the distance traveled on a straight stretch of highway t minutes after 2:30pm. For example, at 2:35pm the distance traveled was 4.2 miles. The average speed of the vehicle *(with units – e.g. grams, gallons, etc.)* from 2:32 to 2:37 is

   <u>0.84 miles per minute (= 50.4 miles per hour)</u> .

Average speed is (distance traveled) ÷ (time elapsed). So we have $\frac{5.9 - 1.7}{5} = 0.84$ miles per minute (multiply by 60 to get miles per hour, if you wish).

5. If $\sin\theta = -\frac{2}{3}$ and θ is in quadrant IV, then

   (a) $\cos\theta = \underline{\frac{\sqrt{5}}{3}}$

   (b) $\tan\theta = \underline{-\frac{2}{\sqrt{5}}}$

   (c) $\sec\theta = \underline{\frac{3}{\sqrt{5}}}$

   (d) $\csc\theta = \underline{-\frac{3}{2}}$

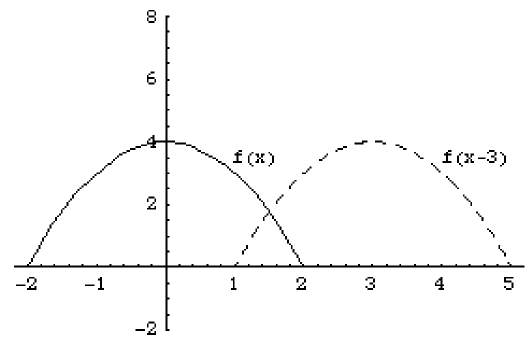
   (e) $\cot\theta = \underline{-\frac{\sqrt{5}}{2}}$

Using a reference angle in quadrant I, we can draw a triangle to figure out what all the trig. functions come out to, up to ± (one leg is 2, the hypotenuse is 3, and so using the Pythagorean Theorem the other leg must be $\sqrt{3^2 - 2^2} = \sqrt{5}$). Then given that θ is in quadrant IV we can put in the correct signs.

Graphs. (20 points)

1. The graph of $f(x)$ is shown. On the same axes, sketch a graph of $f(x-3)$.

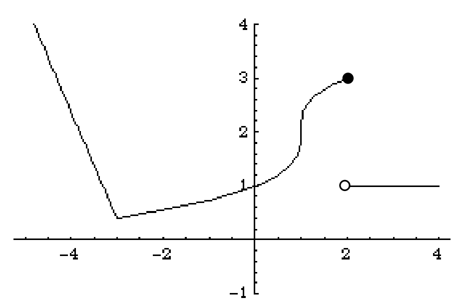
   More accuracy = more points!



2. The graph of the function $f(x)$ shown at right is

   (a) discontinuous at $x =$ _2_ (list all x-values)

   (b) not differentiable (does not have a derivative) at $x =$ _−3, 1, 2_ (list all x-values)



$f(x)$ is continuous everywhere except $x = 2$. But $f(x)$ is not differentiable at $x = -3$ because there is a corner, at $x = 1$ because the tangent line is vertical (and therefore has undefined slope), and at $x = 2$ because of the discontinuity.

Work and Answer. (90 points) *You must show all relevant work to receive full credit.*

1. (15 points) If $f(x) = \sqrt{x^2+1}$ and $g(x) = \dfrac{1}{x^2-1}$, find $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2-1}\right) = \boxed{\sqrt{\left(\frac{1}{x^2-1}\right)^2+1}}$$

2. (15 points) Find the derivative of the function $f(t) = 10t^{18} - \cos t + \tan\left(\frac{1}{t}\right)$.

Remember that the last term here is "tangent **of** $\frac{1}{t}$," so we will be using the chain rule on that term. We have $f'(t) = 180t^{17} + \sin t + \sec^2\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right) = \boxed{180t^{17} + \sin t - \frac{\sec^2\left(\frac{1}{t}\right)}{t^2}}$

3. (15 points) Find the slope of the tangent line to the curve $f(x) = 6xe^{5x^2}$ at the point $(2, 1)$.

The formula for the slope of the tangent line to $f(x)$ is the derivative. Then we plug in $x = 2$ to get the specific slope at that point. Since $f(x)$ is a product, we use the product rule. We have $f'(x) = 6xe^{5x^2}\cdot 10x + e^{5x^2}\cdot 6 = 6e^{5x^2}(10x^2+1)$ (it is easier to use a derivative if you simplify it!), so $f'(2) = 6e^{20}(10\cdot 4+1) = \boxed{246e^{20}}$

4. (15 points) A coin is tossed off the top of a building at the new Mars space station (sometime in the future). Its height in meters after t seconds is $h(t) = 20 - 10t + 1.8t^2$. Find the velocity of the penny after 2 seconds. *You may assume that the penny is still in the air after 2 seconds. Be sure to give units (e.g. feet, kilograms, etc.).*

Velocity is the rate of change (i.e. derivative) of distance. A height is a distance. So we have $v(t) = h'(t) = -10 + 3.6t$. At $t = 2$ the velocity is $v(2) = -10 + 3.6(2) =$ $\boxed{-2.8 \text{ meters per second}}$

(The negative answer means that the penny is on its way down.)

5. (15 points) Suppose

$$\begin{aligned} f(1) &= 5 & g(1) &= 2 \\ f'(1) &= 3 & g'(1) &= 7 \\ f'(2) &= -10 & g'(5) &= -4 \end{aligned}$$

(a) Find $(f \circ g)'(1)$.

Using the chain rule, we have $(f \circ g)'(1) = f'(g(1))g'(1) = f'(2)g'(1) = -10\cdot 7 = \boxed{-70}$

(b) Find $(g \circ f)'(1)$.

Similarly, we have $(g \circ f)'(1) = g'(f(1))f'(1) = g'(5)f'(1) = -4\cdot 3 = \boxed{-12}$

6. (15 points) 1000 bacteria are placed in a jar with enough food so that the population triples every 10 days.

   (a) Write a model for the population t days after the bacteria are placed in the jar.

   Since t represents the number of days, we want to be multiplying by 3 every time t is a multiple of 10. So we have $\boxed{P(t) = 1000 \cdot 3^{t/10}}$

   (b) Suppose the jar is big enough so that in 60 days it will be full. When will the jar be 1/3 full?

   The population triples every 10 days. So ten days before the jar is full, it is 1/3 full. That is, in $\boxed{\text{50 days}}$