Math 75B Selected Homework Solutions

3.7 #6, 10, 28, 33, 41*
16-A #2, 3, 7
4.1 #12, 17, 20, 24, 32, 47, 56*

Completeness:	13	(1 point each problem)
Format:	10	
Total:	23	points
	(+4	possible bonus points)

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§3.7 #33. Find the limit $\lim_{x\to 0}(1-2x)^{1/x}$.

This is an indeterminate form of type $1^{\pm\infty}$, so we use logarithms to help us get the answer. Let $L = \lim_{x\to 0}(1-2x)^{1/x}$. Then

$$\begin{aligned}
\ln(L) &= \ln\left(\lim_{x\to 0}(1-2x)^{1/x}\right) \\
&= \lim_{x\to 0}\ln\left((1-2x)^{1/x}\right) \\
&= \lim_{x\to 0}\frac{1}{x}\ln(1-2x) \\
&= \lim_{x\to 0}\frac{\ln(1-2x)}{x} \qquad \text{“}\frac{0}{0}\text{”} \\
&\overset{\text{H}}{=} \lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1} \\
&= \lim_{x\to 0}\frac{-2}{1-2x} = -2.
\end{aligned}$$

Therefore $L = e^{-2} = \boxed{\dfrac{1}{e^2}}$

§3.7 #41.* If an initial amount A_0 of money is invested at an interest rate r compounded n times per year, the value of the investment after t years is

$$A = A_0\left(1+\frac{r}{n}\right)^{nt}.$$

Show that if interest is compounded continuously (i.e. taking the limit of the above as $n\to\infty$), then the balance after t years is

$$A = A_0e^{rt}.$$

The limit

$$\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

is an indeterminate form of type 1^{∞}, so we use logarithms to help us get the answer. Let $L = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$ (the number A_0 is a constant multiple, so we can put that back on at the end of the problem). Then

$$\begin{aligned}
\ln(L) &= \ln\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\right) \\
&= \lim_{n\to\infty}\left(nt\ln\left(1+\frac{r}{n}\right)\right) \\
&= t\cdot\lim_{n\to\infty} n\ln\left(1+\frac{r}{n}\right) \qquad \text{(as far as the limit is concerned, } t \text{ is a constant!)}
\end{aligned}$$

Now: neat trick! As $n \to \infty$, $\frac{1}{n} \to 0^+$. So we may get the same answer by replacing n with $\dfrac{1}{n}$ and taking the limit as the *new* n goes to 0 from the right:

$$\begin{aligned}
&= t\cdot\lim_{n\to 0^+}\frac{1}{n}\ln(1+rn) \\
&= t\cdot\lim_{n\to 0^+}\frac{\ln(1+rn)}{n} \qquad \text{“}\,\frac{0}{0}\,\text{”} \\
&\overset{\text{H}}{=} t\cdot\lim_{n\to 0^+}\frac{r}{1+rn} \qquad \text{(remember that } r \text{ is constant)} \\
&= t\cdot r = rt.
\end{aligned}$$

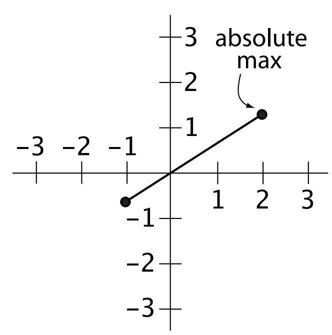
Therefore $L = e^{rt}$, and $A = A_0 e^{rt}$. □

§4.1#12. Sketch a graph of a function defined on the interval $[-1, 2]$ that has

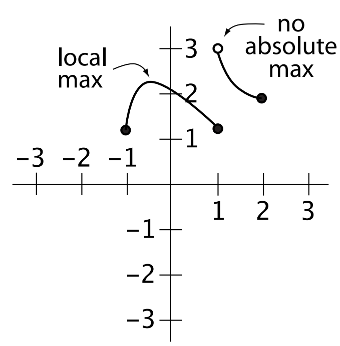
(a) An absolute maximum but no local maximum

(b) A local maximum but no absolute maximum.

(a) There are many possible solutions. Here is one:



(b) We must take care here! The Extreme Value Theorem guarantees that a function defined on a closed interval always has an absolute maximum, *as long as* the function is **continuous** on the interval. So if we want a function defined on $[-1, 2]$ that has NO absolute maximum, we will have to use an example that is **not continuous**. Here is one possible solution:



§4.1#32. Find all critical numbers of $G(x) = \sqrt[3]{x^2 - x}$.

Recall that there are two types of critical numbers:

- Those that, when plugged in, make the derivative equal to zero, and
- Those that make the derivative undefined (but still are in the domain of the function).

The domain of $G(x)$ is all real numbers. So any number that makes $G'(x)$ either 0 or undefined will be a critical number of $G(x)$. We have

$$G'(x) = \frac{1}{3}(x^2 - x)^{-2/3}(2x - 1) = \frac{2x - 1}{3(x^2 - x)^{2/3}}.$$

Taking each type of critical number separately, we have

- $G'(x)$ is equal to zero when the numerator $2x - 1$ is equal to 0. In other words,

$$\begin{aligned} G'(x) = \frac{2x - 1}{3(x^2 - x)^{2/3}} &\overset{\text{set}}{=} 0 \\ 2x - 1 &= 0 \\ x &= \frac{1}{2} \end{aligned}$$

- $G'(x)$ is undefined when the denominator $3(x^2 - x)^{2/3}$ is equal to 0. Solving for x we get

$$\begin{aligned} 3(x^2 - x)^{2/3} &= 0 \\ x^2 - x &= 0 \\ x(x - 1) &= 0 \\ x &= 0,\ x = 1 \end{aligned}$$

Therefore the critical numbers of $G(x)$ are $\boxed{\frac{1}{2},\ 0, \text{ and } 1}$