Math 75B Selected Homework Solutions

Completeness:	16	(1 point each problem)
Format:	10	
Total:	26	points
	(+2	possible bonus points)

17-A #1, 3
17-B #1, 2
4.3 #6, 15, 16, 30, 34
17-C #3, 4, 5
4.4 #2, 10, 38, 40, 46*

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§4.3 #6. For the function $f(x) = x^2 e^x$,

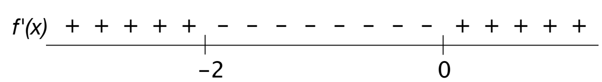
(a) Find the intervals on which f is increasing or decreasing.

The domain of $f(x)$ is all real numbers. We have $f'(x) = x^2e^x + 2xe^x = xe^x(x+2)$, which is defined everywhere. Setting $f'(x)$ equal to 0 we get

$$\begin{aligned} xe^x(x+2) &= 0 \qquad (1)\\ x = 0; \quad x &= -2 \end{aligned}$$

(e^x can never be equal to 0). So we test the intervals $(-\infty, -2)$, $(-2, 0)$ and $(0, \infty)$ using the formula in (1):

$$\begin{aligned} f'(-10) &= (-)(+)(-) = (+)\\ f'(-1) &= (-)(+)(+) = (-)\\ f'(1) &= (+)(+)(+) = (+) \end{aligned}$$



Therefore the function is $\boxed{\text{increasing on the intervals } (-\infty,-2) \text{ and } (0,\infty) \text{ and decreasing on the interval } (-2,0)}$

(b) Find the local maximum and minimum values of f.

From part (a), there is a local maximum of $f(x)$ at $x = -2$ and a local minimum at $x = 0$. Plugging these numbers into $f(x)$ we have that the local maximum value is $f(-2) = (-2)^2e^{-2} = \boxed{\dfrac{4}{e^2}}$ and the local minimum value is $f(0) = 0^2e^0 = \boxed{0}$

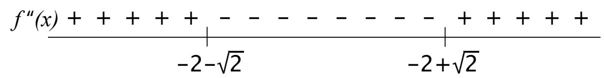
(c) Find the intervals of concavity and the inflection points.

We repeat the above procedure with the second derivative: we have $f''(x) = x^2e^x + 2xe^x + 2xe^x + 2e^x = e^x(x^2 + 4x + 2)$, which is defined everywhere. Setting $f''(x)$ equal to 0 we get

$$\begin{aligned} e^x(x^2 + 4x + 2) &= 0 \qquad (2)\\ x = -2 + \sqrt{2} \approx -0.585; \quad x &= -2 - \sqrt{2} \approx -3.414 \end{aligned}$$

(using the quadratic formula). So we test the intervals $(-\infty, -2-\sqrt{2})$, $(-2-\sqrt{2}, -2+\sqrt{2})$ and $(-2+\sqrt{2}, \infty)$ using the formula in (2):

$$\begin{aligned} f''(-10) &= (+)(+) = (+) \\ f''(-1) &= (+)(-) = (-) \\ f''(0) &= (+)(+) = (+) \end{aligned}$$



Therefore $f(x)$ is $\boxed{\begin{array}{l}\text{concave up on the intervals } (-\infty, -2-\sqrt{2}) \text{ and } (-2+\sqrt{2}, \infty) \\ \text{and concave down on the interval } (-2-\sqrt{2}, -2+\sqrt{2}).\end{array}}$

Since the concavity changes at $x = -2 \pm \sqrt{2}$, both are inflection points — or, more precisely, the inflection *points* are $\boxed{(-2 \pm \sqrt{2}, (-2 \pm \sqrt{2})^2 e^{-2\pm\sqrt{2}})}$

§4.3 #30. For the function $B(x) = 3x^{2/3} - x$,

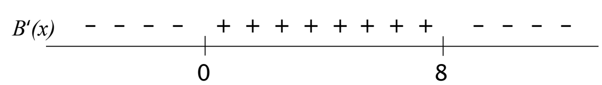
(a) Find the intervals of increase or decrease.

We proceed similar to #6 above: the domain is all real numbers. We have

$$\begin{aligned} B'(x) = 2x^{-1/3} - 1 &\overset{\text{set}}{=} 0 \\ \frac{2}{x^{1/3}} &= 1 \\ x^{1/3} &= 2 \\ x &= 8. \end{aligned}$$

Also the derivative is undefined at $x = 0$, so this is also a critical number.

$$\begin{aligned} B'(-1) &= \frac{2}{-1} - 1 = (-) \\ B'(1) &= \frac{2}{1} - 1 = (+) \\ B'(27) &= \frac{2}{3} - 1 = (-) \end{aligned}$$



Therefore $B(x)$ is $\boxed{\begin{array}{l}\text{increasing on the interval } (0, 8) \\ \text{and decreasing on the intervals } (-\infty, 0) \text{ and } (8, \infty)\end{array}}$

(b) Find the local maximum and minimum values.

From part (a), there is a local minimum of $B(x)$ at $x = 0$ and a local maximum at $x = 8$. Plugging these numbers into $B(x)$ we have that the local minimum value is $B(0) = 0 - 0 = \boxed{0}$ and the local maximum value is $B(8) = 3 \cdot 8^{2/3} - 8 = \boxed{4}$

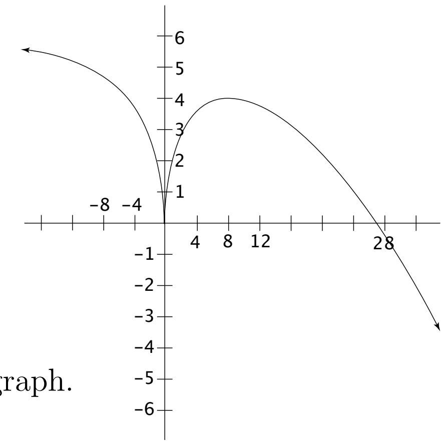
(c) Find the intervals of concavity and the inflection points.

We have $B''(x) = -\frac{2}{3}x^{-4/3} = -\frac{2}{3x^{4/3}}$, which is never equal to 0 and is only undefined at $x = 0$. Therefore we have

$$B''(-1) = -\frac{2}{3} < 0$$
$$B''(1) = -\frac{2}{3} < 0$$

(feel free to draw in your own number line here).
Therefore $B(x)$ is **concave down on the intervals $(-\infty, 0)$ and $(0, \infty)$ and is never concave up.**

Since the concavity does not change at $x = 0$, there is **no inflection point**



(d) Use information from parts (a) - (c) to sketch the graph.

The graph is shown at right.

§4.4#40. Use the guidelines of this section to sketch the curve $f(x) = x(\ln(x))^2$.

Note. For this problem it is useful to remember that $\ln(e^a) = a$. In particular, if a is negative, we get facts like $\ln\left(\frac{1}{e^2}\right) = \ln(e^{-2}) = -2$.

(a) **Domain.** The domain of $f(x)$ is $\boxed{x > 0}$

(b) **Intercepts.** There is no y-intercept, since $f(0)$ is not defined. To get x-intercepts, set $f(x) = 0$:

$$x(\ln(x))^2 = 0$$
$$x = 0 \text{ (not a valid number in the domain);} \quad \ln(x) = 0$$
$$x = 1$$

Therefore the only x-intercept is the point $\boxed{(1, 0)}$

(c) **Symmetry.** Since $f(-x)$ is undefined for $x > 0$, $f(x)$ is neither even nor odd, nor is it periodic.

(d) **Asymptotes.** We have

$$\begin{aligned}
&\lim_{x\to 0^+} x(\ln(x))^2 \qquad \text{(this is an indeterminate form of type } 0\cdot\infty)\\
&= \lim_{x\to 0^+} \frac{(\ln(x))^2}{\frac{1}{x}} \qquad \text{“}\frac{\infty}{\infty}\text{”}\\
&\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{2(\ln(x))\cdot\frac{1}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} -2x\ln(x) \qquad (0\cdot\infty \text{ again})\\
&= \lim_{x\to 0^+} \frac{-2\ln(x)}{\frac{1}{x}} \qquad \text{“}-\frac{\infty}{\infty}\text{”}\\
&\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} 2x = 0.
\end{aligned}$$

Whew! So there will be an open circle at the origin, but $\boxed{\text{no vertical asymptotes}}$

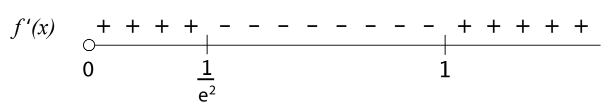
We also have $\lim_{x\to\infty} x(\ln(x))^2 = \infty$, so there will be $\boxed{\text{no horizontal asymptotes}}$

(e) **Increase/Decrease.** We have $f'(x) = 2x\ln(x)\cdot\frac{1}{x} + (\ln(x))^2 = 2\ln(x) + (\ln(x))^2 = \ln(x)(2+\ln(x))$. Setting this equal to 0 we get

$$\begin{aligned}
\ln(x) = 0; &\quad 2+\ln(x) = 0\\
x = 1; &\quad \ln(x) = -2\\
x = 1; &\quad x = e^{-2} = \frac{1}{e^2}
\end{aligned}$$

The derivative is defined for all $x > 0$, so there are no other critical numbers.

$$\begin{aligned}
f'\left(\frac{1}{e^4}\right) &= (-4)(2-4) = (+)\\
f'\left(\frac{1}{e}\right) &= (-1)(2-1) = (-)\\
f'(e) &= (1)(2+1) = (+)
\end{aligned}$$



Therefore the function is $\boxed{\begin{array}{l}\text{increasing on the intervals } \left(0, \frac{1}{e^2}\right) \text{ and } (1,\infty)\\ \text{and decreasing on the interval } \left(\frac{1}{e^2}, 1\right)\end{array}}$

(f) **Local Max/Min Values.** We have $f\left(\frac{1}{e^2}\right) = \frac{1}{e^2}\cdot(\ln(e^{-2}))^2 = \boxed{\frac{4}{e^2}}$ (local max. value) and $f(1) = 1\cdot(\ln(1))^2 = \boxed{0}$ (local min. value)

(g) **Concavity/Inflection Points.** We have $f''(x) = \frac{2}{x} + 2\ln(x) \cdot \frac{1}{x} = \frac{2}{x}(1 + \ln(x))$. Setting this equal to 0 we get

$$1 + \ln(x) = 0$$
$$x = e^{-1} = \frac{1}{e}$$

$$f''\left(\frac{1}{e^2}\right) = (+)(1-2) = (-)$$
$$f'(1) = 2 \cdot 1 = (+)$$

(feel free to draw in your own number line here).

Therefore $f(x)$ is concave down on the interval $\left(0, \frac{1}{e}\right)$ and concave up on the interval $\left(\frac{1}{e}, \infty\right)$.

We have $f\left(\frac{1}{e}\right) = \frac{1}{e}(-1)^2 = \frac{1}{e}$. Since the concavity changes at $x = \frac{1}{e}$, there is an inflection point $\left(\frac{1}{e}, \frac{1}{e}\right)$

(h) **The Graph.**