Math 152 Practice for Midterm III
§§4.9-7.3

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length, format or content, to the actual exam.

If I were you, I would:

- Know all of the definitions mentioned in class and in the sections of the book, and know examples.
- Know all of the theorems mentioned in class and in the sections of the book, and know examples relating to them.
- Go over all of the homework problems, even "redoing" them on WeBWorK in order to practice.
- Go over all quiz problems.
- Especially practice proving things.

You should also know how to do the following:

1. Find the rank and nullity of a matrix
2. Use the Gram-Schmidt process to find an orthonormal basis for a vector space (I will give you the formula
$$\mathbf{v}_i = \mathbf{u}_i - \frac{\mathbf{v}_1 \cdot \mathbf{u}_i}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_i}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \dots - \frac{\mathbf{v}_{i-1} \cdot \mathbf{u}_i}{\mathbf{v}_{i-1} \cdot \mathbf{v}_{i-1}}\mathbf{v}_{i-1};$$
however, it is up to you to know how to use it!)
3. Prove or disprove that a given function is a linear transformation
4. Find the eigenvalues and associated eigenvectors of a square matrix
5. Prove or disprove that a matrix is diagonalizable
6. Find the diagonalization matrix of a diagonalizable matrix A (i.e. find a matrix P such that $P^{-1}AP$ is diagonal)
7. Use diagonalization to compute a power of a matrix, for example A^{152}.
8. For a given linear transformation $L\colon V \to W$,
   (a) Compute the kernel of L
   (b) Compute the image of L
   (c) Determine whether or not L is one-to-one, onto, invertible, an isomorphism
   (d) Determine the inverse of an invertible linear transformation
9. Know the definition of a *matrix of a transformation*. Know how to compute it.

Here are some sample problems:

1. Let $L\colon P_2 \to P_3$ be defined by $L(p(t)) = t^2 p'(t)$.
   (a) Prove that L is a linear transformation.
   (b) Prove or disprove: L is one-to-one. If L is not one-to-one, find
      i. $\ker(L)$
      ii. A basis for $\ker(L)$
      iii. The dimension of $\ker(L)$.
   (c) Prove or disprove: L is onto. If L is not onto, find
      i. $\mathrm{im}(L)$
      ii. A basis for $\mathrm{im}(L)$
      iii. The dimension of $\mathrm{im}(L)$.

2. **Prove or Disprove.** *If the statement is true, prove it. Use definitions and theorems. If the statement is false, give a counterexample.*
   (a) If the rank of an $n \times n$ matrix A is n, then A is invertible.
   (b) Let $S = \{\mathbf{u}_1, \mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $\mathbf{u}$ is orthogonal to every vector in S, then $\mathbf{u}$ is orthogonal to every vector in $\mathrm{span}(S)$.
   (c) Let $L\colon V \to W$ be a linear transformation, and let T be a subspace of W. Then the set
   $$S = \{\mathbf{v} \in V \mid L(\mathbf{v}) \in T\}$$
   is a subspace of V.
   (d) A linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if the matrix of L is invertible.
   (e) If λ is an eigenvalue of a matrix A with eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k with eigenvector $\mathbf{x}$.
   (f) If λ is an eigenvalue of an invertible matrix A with eigenvector $\mathbf{x}$, then $-\lambda$ is an eigenvalue of A^{-1} with eigenvector $-\mathbf{x}$.
   (g) If a matrix $A_{n\times n}$ has row k equal to the kth row of I_n for some k, then 1 is an eigenvalue of A.
   (h) If A is an $n \times n$ matrix and the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} = \mathbf{u}$, then $\mathbf{u}$ is an eigenvector of A.
   (i) If A and B are invertible $n \times n$ matrices, then AB^{-1} and BA^{-1} have the same eigenvalues.

3. Find the matrix of the linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L\left(\begin{bmatrix}-1\\3\end{bmatrix}\right) = \begin{bmatrix}3\\-7\\-2\end{bmatrix}$ and $L\left(\begin{bmatrix}4\\2\end{bmatrix}\right) = \begin{bmatrix}-6\\1\\1\end{bmatrix}$.

4. Let V and W be vector spaces with $\dim V = 3$ and $\dim W = 4$, and let $L\colon V \to W$ be a linear transformation. Which of the following scenarios are possible? For each part, if it is possible, give an example. If it is not possible, explain why not.

   (a) L is one-to-one.

   (b) L is onto.

   (c) L is one-to-one, but not onto.

   (d) L is onto, but not one-to-one.

   (e) L is both one-to-one and onto.

5. Repeat #4 for $\dim V = 4$ and $\dim W = 3$.

6. Repeat #4 for $\dim V = 4$ and $\dim W = 4$.