then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus, $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 Exercises

1–80 Evaluate the integral.

1. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx$
2. $\displaystyle\int \tan^3\theta\,d\theta$
3. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt$
4. $\displaystyle\int \frac{x}{\sqrt{3-x^4}}\,dx$
5. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$
6. $\displaystyle\int x \csc x \cot x\,dx$
7. $\displaystyle\int_1^3 r^4 \ln r\,dr$
8. $\displaystyle\int_0^4 \frac{x-1}{x^2-4x-5}\,dx$
9. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx$
10. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx$
11. $\displaystyle\int \sin^3\theta \cos^5\theta\,d\theta$
12. $\displaystyle\int \sin x \cos(\cos x)\,dx$
13. $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}}$
14. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x \ln x}\,dx$
15. $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$
16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
17. $\displaystyle\int x \sin^2 x\,dx$
18. $\displaystyle\int \frac{e^{2t}}{1+e^{4t}}\,dt$
19. $\displaystyle\int e^{x+e^x}\,dx$
20. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$
21. $\displaystyle\int t^3 e^{-2t}\,dt$
22. $\displaystyle\int x \sin^{-1}x\,dx$
23. $\displaystyle\int_0^1 (1+\sqrt{x})^8\,dx$
24. $\displaystyle\int \ln(x^2-1)\,dx$
25. $\displaystyle\int \frac{3x^2-2}{x^2-2x-8}\,dx$
26. $\displaystyle\int \frac{3x^2-2}{x^3-2x-8}\,dx$
27. $\displaystyle\int \cot x \ln(\sin x)\,dx$
28. $\displaystyle\int \sin\sqrt{at}\,dt$
29. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw$
30. $\displaystyle\int_{-2}^2 |x^2-4x|\,dx$
31. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$
32. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx$
33. $\displaystyle\int \sqrt{3-2x-x^2}\,dx$
34. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$
35. $\displaystyle\int_{-1}^1 x^8 \sin x\,dx$
36. $\displaystyle\int \sin 4x \cos 3x\,dx$
37. $\displaystyle\int_0^{\pi/4} \cos^2\theta \tan^2\theta\,d\theta$
38. $\displaystyle\int_0^{\pi/4} \tan^5\theta \sec^3\theta\,d\theta$
39. $\displaystyle\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$
40. $\displaystyle\int \frac{1}{\sqrt{4y^2-4y-3}}\,dy$
41. $\displaystyle\int \theta \tan^2\theta\,d\theta$
42. $\displaystyle\int x^2 \tan^{-1}x\,dx$
43. $\displaystyle\int e^x\sqrt{1+e^x}\,dx$
44. $\displaystyle\int \sqrt{1+e^x}\,dx$
45. $\displaystyle\int x^5 e^{-x^3}\,dx$
46. $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$
47. $\displaystyle\int \frac{x+a}{x^2+a^2}\,dx$
48. $\displaystyle\int \frac{x}{x^4-a^4}\,dx$

49. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$

50. $\displaystyle\int \frac{1}{x^2\sqrt{4x+1}}\,dx$

51. $\displaystyle\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

52. $\displaystyle\int \frac{dx}{x(x^4+1)}$

53. $\displaystyle\int x^2 \sinh mx\,dx$

54. $\displaystyle\int (x+\sin x)^2\,dx$

55. $\displaystyle\int \frac{1}{x+4+4\sqrt{x+1}}\,dx$

56. $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

57. $\displaystyle\int x\sqrt[3]{x+c}\,dx$

58. $\displaystyle\int x^2\ln(1+x)\,dx$

59. $\displaystyle\int \frac{1}{e^{3x}-e^x}\,dx$

60. $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$

61. $\displaystyle\int \frac{x^4}{x^{10}+16}\,dx$

62. $\displaystyle\int \frac{x^3}{(x+1)^{10}}\,dx$

63. $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

64. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

65. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

hint

66. $\displaystyle\int_2^3 \frac{u^3+1}{u^3-u^2}\,du$

67. $\displaystyle\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt$

68. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$

69. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

70. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$

71. $\displaystyle\int \frac{x}{x^4+4x^2+3}\,dx$

72. $\displaystyle\int \frac{\sqrt{t}}{1+\sqrt[3]{t}}\,dt$

73. $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,dx$

74. $\displaystyle\int \frac{dx}{e^x-e^{-x}}$

75. $\displaystyle\int \sin x\sin 2x\sin 3x\,dx$

76. $\displaystyle\int (x^2-bx)\sin 2x\,dx$

77. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

78. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

79. $\displaystyle\int x\sin^2 x\cos x\,dx$

80. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

81. The functions $y = e^{x^2}$ and $y = x^2e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2+1)e^{x^2}$ does. Evaluate $\int (2x^2+1)e^{x^2}\,dx$.

7.6 Integration Using Tables and Computer Algebra Systems

In this section we describe how to use tables and computer algebra systems to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even the most powerful computer algebra systems can't find explicit formulas for the antiderivatives of functions like e^{x^2} or the other functions described at the end of Section 7.5.

Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don't have access to a computer algebra system. A relatively brief table of 120 integrals, categorized by form, is provided on the Reference Pages at the back of the book. More extensive tables are available in *CRC Standard Mathematical Tables and Formulae,* 30th ed. by Daniel Zwillinger (Boca Raton, FL: CRC Press, 1995) (581 entries) or in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products,* 6e (New York: Academic Press, 2000), which contains hundreds of pages of integrals. It should be remembered, however, that integrals do not often occur in exactly the form listed in a table. Usually we need to use substitution or algebraic manipulation to transform a given integral into one of the forms in the table.

EXAMPLE 1 The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

SOLUTION Using the method of cylindrical shells, we see that the volume is

$$V = \int_0^1 2\pi x\arctan x\,dx$$