Worksheet – Techniques of Integration Necessary for Section 6.3

1. $\displaystyle\int \frac{1}{2x-1}\,dx.$

   Hint. Let $u = 2x - 1$.

   Let $u = 2x - 1$. Then $du = 2\,dx$, and we get

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|2x-1| + C}$$

   Moral. You can integrate anything that looks like $\dfrac{\text{constant}}{\text{linear}}$!

2. $\displaystyle\int \frac{2x-5}{(x^2-5x)^3}\,dx.$

   Hint. Let $u = x^2 - 5x$.

   Let $u = x^2 - 5x$. Then $du = 2x - 5\,dx$, and we get

$$\int \frac{1}{u^3}\,du = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C = \boxed{-\frac{1}{2(2x-5)^2} + C}$$

   Moral. Always check to see if you can use u-substitution before trying anything fancy!

3. $\displaystyle\int \frac{4x-1}{x^2+5}\,dx.$

   Hint. Split up the fraction, then use u-substitution (with $u = x^2 + 5$) on one term and the following **formula** on the other:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

   We have

$$\int \frac{4x-1}{x^2+5}\,dx = \int \frac{4x}{x^2+5}\,dx - \int \frac{1}{x^2+5}\,dx.$$

   For the first term, let $u = x^2 + 5$. Then $du = 2x\,dx$, and the integral becomes $2\displaystyle\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln|x^2+5| + C$. For the second term, use the **formula** with $a = \sqrt{5}$. So the answer to the original integral becomes

$$\boxed{2\ln|x^2+5| + \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C}$$

   Moral: You can integrate anything that looks like $\dfrac{\text{linear}}{x^2+a^2}$!

4. $\displaystyle\int \frac{3x-1}{x^2+6x+11}\,dx.$

Hint. Complete the square in the denominator, *i.e.* $x^2+6x+11 = x^2+6x+9+2 = (x+3)^2+2$. Then let $u = x+3$, and apply the technique in problem 3, above.

If $u = x+3$ then $x = u-3$. $du = dx$, so completing the square as in the hint, we have

$$\begin{aligned}\int \frac{3x-1}{x^2+6x+11}\,dx &= \int \frac{3x-1}{(x+3)^2+2}\,dx \\ &= \int \frac{3(u-3)-1}{u^2+2}\,du \\ &= \int \frac{3u-10}{u^2+2}\,du \\ &= 3\int \frac{u}{u^2+2}\,du - 10\int \frac{1}{u^2+2}\,du.\end{aligned}$$

Now this looks just like the previous problem. Use u-substitution (or choose a different letter, since we're already in u) for the first term, and the inverse-tangent formula for the second; we get

$$\begin{aligned}\frac{3}{2}\ln|u^2+2| - \frac{10}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C &= \frac{3}{2}\ln|(x+3)^2+2| - \frac{10}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C \\ &= \boxed{\frac{3}{2}\ln|x^2+6x+11| - \frac{10}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C}\end{aligned}$$

Moral: You can integrate anything that looks like $\dfrac{\text{linear}}{\text{quadratic}}$!

5. $\displaystyle\int \frac{x^3-3x^2+1}{x^2+1}\,dx.$

Hint. Perform **polynomial division**.

Check to make sure you get $x-3+\dfrac{-x+4}{x^2+1}$. Integrate.

We verified this long division in class. Now we have

$$\begin{aligned}\int \frac{x^3-3x^2+1}{x^2+1}\,dx &= \int \left(x-3+\frac{-x+4}{x^2+1}\right)\,dx \\ &= \frac{1}{2}x^2 - 3x - \int \frac{x}{x^2+1}\,dx + 4\int \frac{1}{x^2+1}\,dx \\ &= \boxed{\frac{1}{2}x^2 - 3x - \frac{1}{2}\ln|x^2+1| + 4\tan^{-1}x + C}\end{aligned}$$

Moral. When the integrand is an *improper* rational function, perform polynomial division to rewrite the quotient as a polynomial plus a *proper* rational function, then apply the previous techniques.