Section 6.2 - Trigonometric Integrals Worksheet – Solutions

Recall that the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

can be used to evaluate integrals of the form $\int \sin^m x \cos^n x \, dx$ as long as either m or n is odd. Practice this technique with the following integral:

1. $\int \sin^5 x \cos^2 x \, dx$

Since the power of $\sin x$ is odd, let $u = \cos x$. Then $du = -\sin x \, dx$, and we have

$$\begin{aligned}
&\int \sin^5 x \cos^2 x \, dx \\
&= \int \sin^4 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx \\
&= -\int (1 - u^2)^2 u^2 \, du \\
&= -\int (1 - 2u^2 + u^4) u^2 \, du \\
&= -\int (u^2 - 2u^4 + u^6) \, du \\
&= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \\
&= \boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}
\end{aligned}$$

Now we will develop a similar strategy for integrals of the form $\int \tan^m x \sec^n x \, dx$. What do you get when you divide both sides of the Pythagorean identity by $\cos^2 x$? Simplify your answer and write the new identity here:

$$\underline{\tan^2 x + 1 = \sec^2 x}$$

(it should be in terms of $\tan x$ and $\sec x$).

Now try the following integrals. Work with your group to develop a strategy for using the new identity (also called a Pythagorean identity since it comes directly from the other one) to solve these: [1]

2. $\int \tan^6 x \sec^2 x \, dx$

Here we don't need any identities. We simply let $u = \tan x$, so that $du = \sec^2 x \, dx$, and we have

$$\begin{aligned}
\int \tan^6 x \sec^2 x \, dx &= \int u^6 \, du \\
&= \frac{1}{7}u^7 + C \\
&= \boxed{\frac{1}{7}\tan^7 x + C}
\end{aligned}$$

over for more fun!

[1] Hint. You may want to recall the derivatives of $\tan x$ and of $\sec x$ before you begin.

$(\tan x)' = \underline{\sec^2 x}$ $(\sec x)' = \underline{\sec x \tan x}$

Write your new identity again here, for reference:

$\underline{\tan^2 x + 1 = \sec^2 x}$

3. $\displaystyle\int \tan^2 x \sec^6 x \, dx$

The power of $\sec x$ is even, so we can let $du = \sec^2 x \, dx$ (i.e. let $u = \tan x$) and still be left with an even power of $\sec x$ on which to use the Pythagorean identity. We have

$$\begin{aligned}
&\int \tan^2 x \sec^6 x \, dx \\
&= \int \tan^2 x \sec^4 x \sec^2 x \, dx \\
&= \int \tan^2 x (\tan^2 x + 1)^2 \sec^2 x \, dx \\
&= \int u^2 (u^2 + 1)^2 \, du
\end{aligned}$$

$$\begin{aligned}
&= \int u^2 (u^4 + 2u^2 + 1) \, du \\
&= \int (u^6 + 2u^4 + u^2) \, du \\
&= \frac{1}{7}u^7 + \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \boxed{\frac{1}{7}\tan^7 x + \frac{2}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C}
\end{aligned}$$

4. $\displaystyle\int \tan^3 x \sec x \, dx$

Here the above strategy does not work, since the power of $\sec x$ is odd. However, since the power of $\tan x$ is odd, we can let $du = \sec x \tan x \, dx$ (i.e. let $u = \sec x$ and still be left with an even power of $\tan x$ on which to use the Pythagorean identity. We have

$$\begin{aligned}
&\int \tan^3 x \sec x \, dx \\
&= \int \tan^2 x \sec x \tan x \, dx \\
&= \int (\sec^2 x - 1) \sec x \tan x \, dx
\end{aligned}$$

$$\begin{aligned}
&= \int (u^2 - 1) \, du \\
&= \int \frac{1}{3}u^3 - u + C \\
&= \boxed{\frac{1}{3}\sec^3 x - \sec x + C}
\end{aligned}$$

Can you think of situations where your strategy will not work for integrals like $\displaystyle\int \tan^m x \sec^n x \, dx$?

In #3, the power of $\sec x$ is even. In this case our strategy was to let $u = \tan x$ and use the Pythagorean identity when needed to rewrite all but two of the powers of $\sec x$ in terms of $\tan x$.

In #4, the power of $\tan x$ is odd. In this case our strategy was to let $u = \sec x$ and use the Pythagorean identity when needed to rewrite all but two of the powers of $\tan x$ in terms of $\sec x$.

This strategy will not work if the power of $\sec x$ is odd *and* the power of $\tan x$ is even. It also will not work if the power of $\sec x$ is 0 (i.e. if you have an integral of the form $\displaystyle\int \tan^m x \, dx$), or if the power of $\tan x$ is 0 (i.e. if you have an integral of the form $\displaystyle\int \sec^n x \, dx$). Can you think of ways to modify the above strategies to fit these cases?