Embedding Graphs into Finite Projective Planes

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Graphs

- Graphs will be considered to be finite, simple and undirected.

- We write $G = (V, E)$ to denote that $V$ and $E$ are the set of vertices and edges of $G$, respectively.

We will focus mostly on cycles and complete bipartite graphs.

$C_6$

$K_{3,3}$
Finite Projective Planes

- A projective plane $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and an incidence relation $\mathcal{I}$ such that:
  (a) Any two points determine a unique line. Any two lines intersect in exactly one point.
  (b) Lines contain at least three points.
  (c) There are four points, no three of them on a line.

- Let $\pi$ be a finite projective plane, then there is a positive integer $q$, called the order of $\pi$, such that $|\mathcal{P}| = |\mathcal{L}| = t_q = q^2 + q + 1$. Also, every point is on exactly $q + 1$ lines and every line contains exactly $q + 1$ points.
Definitions: Embeddings and Levi Graphs

**Definition**

Let $G$ be a graph and let $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite projective plane. An embedding of $G$ into $\pi$ is an injective function

$$\epsilon : V(G) \to \mathcal{P}$$

that, by preserving incidence, yields an injective function

$$\bar{\epsilon} : E(G) \to \mathcal{L}$$

If $\epsilon$ is an embedding of $G$ into $\pi$ then we write $\epsilon : G \hookrightarrow \pi$.

**Definition**

The Levi graph, $Levi(\pi)$, of a plane $\pi$ is a bipartite graph with vertex set $\mathcal{P} \cup \mathcal{L}$, and edges joining points and lines that are incident in $\pi$. 
Examples

Three embeddings of $K_4$ in $PG(2, 2)$

$Levi(PG(2, 2))$ is the Heawood graph\(^1\)

\(^1\)http://www.hindawi.com/journals/ijcom/2010/767361/
• Erdős (1979) said that embeddings of linear spaces into finite projective planes was interesting.

• Want a Kuratowski-type result for graphs in \( \pi \).

• Want to study \( \pi \) by understanding what types of graphs embed in \( \pi \), and how many embeddings of such graph into \( \pi \) there are.

• Use geometric methods to study what cycles are contained in \( \text{Levi}(\pi) \), where \( \pi \) is a finite projective plane.
Why Levi Graphs?

*Levi*($\pi$) is an extremal object.

**Definition**

For every $q \geq 2$, let $\mathcal{F}_q$ be the family of all 4-cycle-free $t_q$ by $t_q$ bipartite graphs.

- *Levi*($\pi$) has the largest number of edges among all graphs in $\mathcal{F}_q$.
- Every *Levi*($\pi$) has the largest number of 6-cycles among all graphs in $\mathcal{F}_q$ and, if $q \geq 157$, the greatest number of 8-cycles.
- It has been conjectured that, if $q$ is large compared to $k$, *Levi*($\pi$) is the graph in $\mathcal{F}_q$ containing the greatest number of $2k$-cycles.

**Remark**

The number of $2k$-cycles in *Levi*($\pi$) is equal to the number of $k$-cycles embedded in $\pi$. 
Counting Embeddings

Definition
Let $\phi, \psi$ be embeddings of a graph $G$ into $\pi$.

• If $\psi = \phi \circ \varphi$, for some $\varphi \in \text{Aut}(G)$, we will say that $\phi$ and $\psi$ are equivalent.

• $N_{\pi}(G)$ is the number of embeddings of $G$ in $\pi$.

• $n_{\pi}(G)$ is the number of un-equivalent embeddings of $G$ into $\pi$.

Theorem (Mellinger-Vaughn-V. 2013)
Assume $G \hookrightarrow \pi$. Then, $N_{\pi}(G) = n_{\pi}(G)|\text{Aut}(G)|$. 
How Many Cycles in $\pi$?

Theorem (Lazebnik-Mellinger-V. 2009)

Let $C_k$ be a $k$-cycle, and $\pi$ a projective plane of order $q \geq 2$. Then,

- $n_\pi(C_3) = \frac{1}{6} q^2 t_q(t_q - 1)$.
- $n_\pi(C_4) = \frac{1}{8} q^2 t_q(t_q - 1)(q - 1)^2$.
- $n_\pi(C_5) = \frac{1}{10} q^2 t_q(t_q - 1)(q - 1)^2(t_q - q)$.
- $n_\pi(C_6) = \frac{1}{12} q^2 t_q(t_q - 1)(q - 1)^2(q + 2)(q^3 - 2q^2 - q + 3)$.

Voropaev (2013) extended this list to $k \leq 10$. He also proved:

$$n_\pi(C_k) = \frac{q^{2k}}{2k} + O(q^{2k-2})$$
Definition

A finite affine/projective plane $\pi$ is pancyclic if every possible cycle may be embedded in $\pi$.

Theorem (Lazebnik-Mellinger-V. 2013)

All finite affine/projective planes are pancyclic.

Proof.
Complete Bipartite Graphs

Theorem (Mellinger-Vaughn-V. 2013)

If a complete bipartite graph embeds in $\pi$ (of order $q$) then:

- $K_{n,m}$, where $1 \leq n, m \leq q$, or
- $K_{1,q+1}$.

Theorem (Mellinger-Vaughn-V. 2013)

Consider $\epsilon : K_{n,q} \hookrightarrow \pi$, for $2 \leq n \leq q$. If the vertices of $\epsilon(K_{n,q})$ always lay on two lines, then:

- For $n = q$ we get: $n_{\pi}(K_{q,q}) = \binom{t_q}{2}$
- For $n < q$ we get

$$n_{\pi}(K_{n,q}) = 2 \binom{t_q}{2} \binom{q}{n}$$

If $n = q$, or if $n < q$ and $(q - n)^2 < q$, then the hypothesis are satisfied.
We have constructed examples of embeddings using

- Baer subplanes.
- Ovals/hyperovals.

The existence/number of these objects does not depend solely on the order of a plane, but on its structure.

Classification using embeddings?

\textbf{Theorem (Mellinger-Vaughn-V. 2013)}

\textit{Every graph can be embedded in a large enough projective plane.}

Classification of planes possible using minimality among planes admitting the embedding of a certain graph?
Thanks!