Feet in Buekenhout-Metz Unitals

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- $\mathbb{F}_{q^2}$ is the field of order $q^2$, where $q = p^n$ and $p$ is an odd prime.

$$\mathbb{F}_{q^2} = \{x_1 + x_2 \epsilon; \ x_1, x_2 \in \mathbb{F}_q\}$$

where $\epsilon \in \mathbb{F}_{q^2}$ is such that $w = \epsilon^2$ is a non-square in $\mathbb{F}_q$ and $\epsilon^q = -\epsilon$.

- $x_1 + x_2 \epsilon = (x_1 + x_2 \epsilon)^q = x_1 - x_2 \epsilon$

- $T(x) = x + \overline{x}$

  The trace is an additive function.

- $N(x) = x \overline{x}$

  The norm is a multiplicative function.
Let $V = (\mathbb{F}_q^2)^3$. We define the projective plane $PG(2, q^2)$ by:

Points $=$ 1-dimensional subspaces of $V$.
Lines $=$ 2-dimensional subspaces of $V$.
The incidence is given by natural set-theoretic containment.

- A point $P$ of $PG(2, q^2)$ will be denoted by

  $$P = [a, b, c]$$

  where $(a, b, c)$ is a vector generating the subspace defining $P$.

- A line $l$ of $PG(2, q^2)$ will be denoted by

  $$l = [x, y, z]^t$$

  where $(x, y, z)$ is orthogonal to the subspace defining $l$.

- Standard dot product may be used to check incidence.
Unitals in $\text{PG}(2, q^2)$

Definition

A unital in $\text{PG}(2, q^2)$ is a set $U$ of $q^3 + 1$ points such that every line intersects $U$ in exactly 1 or $q + 1$ points.

Definition

For $\alpha, \beta \in \mathbb{F}_{q^2}$ such that $4N(\alpha) + (\overline{\beta} - \beta)^2$ is a non-square in $\mathbb{F}_q$,

$$U_{\alpha, \beta} = \{[x, \alpha x^2 + \beta N(x) + r, 1]; \ x \in \mathbb{F}_{q^2}, \ r \in \mathbb{F}_q\} \cup \{[0, 1, 0]\}$$

is an Orthogonal-Buekenhout-Metz Unital in $\text{PG}(2, q^2)$.

The cases when $\alpha = 0$ (classical) and when $\beta = \overline{\beta}$ (union of conics) are well-understood.
Elementary counting shows:

If \( Q \in U_{\alpha,\beta} \) then there is exactly one tangent line to \( U_{\alpha,\beta} \) through \( Q \).
Feet of a point

Definition
Let $U_{\alpha,\beta}$ be an Orthogonal-Buekenhout-Metz Unital and $P \notin U_{\alpha,\beta}$. Each of the $q + 1$ points of $U_{\alpha,\beta}$ that are on a tangent line to $U_{\alpha,\beta}$ through $P$ is said to be a foot of $P$. We will denote the set of feet of $P$ by $\tau_P(U_{\alpha,\beta})$. 
Properties of $\tau_P(U_{\alpha,\beta})$

$\tau_P(U_{\alpha,\beta})$ has the following properties:

(1) If $U_{\alpha,\beta}$ is classical (i.e. $\alpha = 0$) then the points of $\tau_P(U_{\alpha,\beta})$ are collinear, for all $P \notin U_{\alpha,\beta}$.

(2) If $U_{\alpha,\beta}$ is non-classical, and $\tau_P(U_{\alpha,\beta})$ is contained in a line, then $P \in \ell_\infty$

(3) If $U_{\alpha,\beta}$ is non-classical with $\beta = \overline{\beta}$. If $\tau_P(U_{\alpha,\beta})$ is not contained on a line, then it is contained in an arc.

Our problem is to study $\tau_P(U_{\alpha,\beta})$ for when $\alpha \neq 0$ and $\beta \neq \overline{\beta}$.
The unital $U_{\alpha,\beta}$ admits a group acting on its points in such a way that we can focus on looking only at $\tau_{P_{\lambda}}(U_{\alpha,\beta})$ for $P_{\lambda} = [0, \lambda \epsilon, 1]$, with $\lambda = 1$ or $\lambda = w$.

**Theorem**

$$\tau_{P_{\lambda}}(U_{\alpha,\beta}) = \left\{ Q_x; \; x \in \mathbb{F}_{q^2}, \; 2 \lambda \epsilon + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0 \right\}$$

where

$$Q_x = [x, T(\alpha x^2) - \lambda \epsilon, 1] \quad \text{and} \quad M_{\alpha,\beta} = \begin{bmatrix} \alpha & \frac{1}{2} (\beta - \overline{\beta}) \\ \frac{1}{2} (\beta - \overline{\beta}) & -\overline{\alpha} \end{bmatrix}$$
Geometry of $\tau_{P\lambda}(U_{\alpha,\beta})$

Let $l_{x,y}$ be the line through $Q_x$ and $Q_y$, both points on $\tau_{P\lambda}(U_{\alpha,\beta})$.

**Theorem**

(1) If $\ell$ is a line of $PG(2, q^2)$ then $\ell$ intersects $\tau_{P\lambda}(U_{\alpha,\beta})$ in 0, 1, 2, or 4 points.

(2) $\tau_{P\lambda}(U_{\alpha,\beta})$ can be partitioned by using lines of the form $l_x, -x$ (they intersect $\tau_{P\lambda}(U_{\alpha,\beta})$ in two or four points).
Thank you