1. Let $G$ be a nonempty set and $*: G \times G \to G$ a function such that
   
   (a) $\forall a, b, c \in G : (a * b) * c = a * (b * c)$
   
   (b) $\exists e \in G : \forall a \in G : a * e = a = e * a$
   
   (c) $\forall a \in G : \exists b \in G : a * b = e$

   Prove the following:
   
   • The element $e$ from (b) is unique. So prove: if $e' \in G$ such that $a * e' = a = e' * a$ for all $a \in G$ then $e' = e$.
   
   • If $a, b \in G$ with $a * b = e$ then $b * a = e$.
   
   • Given $a \in G$, the element $b$ from (c) is unique. So prove: if $a, b, c \in G$ with $a * b = e = a * c$ then $b = c$.

2. For a real number $x$, we denote by $[x]$ the largest integer smaller than or equal to $x$.

   Let $G = \{x \in \mathbb{R} : 0 \leq x < 1\}$. For $x, y \in G$, we define
   
   $$ x * y = x + y - [x + y] $$

   Prove that $*$ is a binary operation (so $x * y \in G$ for all $x, y \in G$) and that $(G, *)$ is an abelian group.

   Clearly explain which properties you are using to prove this.

3. Let $G$ be a group. Prove that $(g^{-1})^{-1} = g$ for all $g \in G$.

4. Let $G$ be a group such that $g^{-1} = g$ for all $g \in G$. Prove that $G$ is abelian.

5. Prove the Subgroup Test:

   Let $G$ be a group and $\emptyset \neq H \subseteq G$. Then $H \leq G$ if and only if $ab^{-1} \in H$ for all $a, b \in H$.

6. Find all the non-trivial subgroups of $Q_8$ and $A_4$ (a subgroup $H$ of a group $G$ is non-trivial if $\{e\} \neq H \neq G$). Provide a table of multiplication for the non-cyclic subgroups.

7. Let $G$ be a group, $I$ an index set and $H_i \leq G$ for all $i \in I$. Prove that $\cap_{i \in I} H_i \leq G$.

8. Rewrite the following elements of $D_6$ in the form $sr^i$ with $0 \leq i \leq 5$:

   (a) $sr^{-2}sr^5$
   
   (b) $r^{-3}sr^4sr^{-2}$

9. Put $H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R}; a \neq 0, c \neq 0 \right\}$. Prove that $H \leq GL(2, \mathbb{R})$.

10. Put $H = \{e, (124), (142)\}$.

    (a) Write down all the left cosets of $H$ in $A_4$.

    (b) Find a left transversal to $H$ in $A_4$. 