1. Prove Proposition 1.18(d)(e)(f).

2. This exercise why we consider series with a countable number of terms.
   
   So let $I$ be an uncountable index set and let $x_i > 0$ for all $i \in I$.
   
   We define
   
   $$ \sum_{i \in I} x_i = \sup_{J \subseteq I, \text{finite}} \sum_{j \in J} x_j $$
   
   Prove that $\sum_{i \in I} x_i = +\infty$.
   
   Hint: Show that $I = \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \}$.

3. Let $x \in (0,1)$ and $p \in \mathbb{N}$ with $p \geq 2$. In this exercise, you will prove that there exists a sequence of integers
   
   $\{a_n\}_{n \geq 1}$ such that $0 \leq a_n \leq p - 1$ for all $n \geq 1$ and $x = \sum_{n=1}^{+\infty} \frac{a_n}{p^n}$. So in base $p$, $x = 0.a_1a_2a_3\ldots$
   
   First, you have to come up with a recursive definition of $a_n$. To make your definitions look nicer, define $a_0 = 0$.
   
   Recall the following definition:
   
   The integral part of a real number $t$ (notation: $[t]$) is the largest integer smaller than or equal to $t$.
   
   Then we have:
   
   If $t \in \mathbb{R}$ and $a \in \mathbb{Z}$ then $a = [t] \iff a \leq t < a + 1$.
   
   To come up with a formula for $a_n$, consider this: if (in decimal form) $x = 0.7\ldots$ then we have/expect
   
   $$ \frac{7}{10} \leq x < \frac{8}{10} $$
   
   We can rewrite this as $7 \leq 10x < 8$.
   
   So in general (in decimal form), if $x = 0.a_1a_2\ldots$ then (it seems) we get that
   
   $$ \frac{a_1}{10} \leq x < \frac{a_1 + 1}{10} $$
   
   and so $a_1 \leq 10x < a_1 + 1$. This should allow you to find a ‘formula’ for $a_1$ in terms of $x$ (or $x$ and $a_0$ since we put $a_0 = 0$) (and yes, it will involve the integral part function).
   
   Now we can do something similar to define $a_2$ in terms of $x$, $a_0$ and $a_1$.
   
   (a) Give a recursive definition for $a_n$ for $n \geq 1$.
   
   (b) Prove that $0 \leq a_n \leq p - 1$ and $0 \leq x - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_n}{p^n} < \frac{1}{p^n}$ for all $n \in \mathbb{N}$.
   
   (c) Prove that $x = \sum_{n=1}^{+\infty} \frac{a_n}{p^n}$. 