

# An $sl(2)$ tangle homology and seamed cobordisms

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We construct a tangle homology theory which depends on a parameter  $a$ , and which is properly functorial under tangle cobordisms.

- For the case of links, and for  $a = 0$ , this construction is isomorphic to the original Khovanov homology theory. In particular, it naturally fixes the sign ambiguity under link cobordisms of the mentioned homology theory.
- For  $a = 1$ , we also recover Lee's variant of the original Khovanov homology theory.
- Since a 2-knot in  $\mathbb{R}^4$  is a cobordism from the empty link to itself, our theory gives an invariant of 2-knots.
- Moreover, this construction corresponds to a modified version of the Khovanov-Rozansky homology theory with  $n = 2$ , where the potential  $x^3$  is replaced by  $x^3 - 3ax$ .

## $sl(2)$ quantum web space

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- It is an approach to the  $sl(2)$  quantum link invariant (thus, it is another way of talking about the Jones polynomial), via a calculus of planar bivalent graphs, called “webs”.
- A **web** (with boundary) is denoted by an oriented graph in a disk with bivalent and univalent vertices, and possibly closed loops, such that the edges are either all “in” or all “out” at the bivalent vertices, and such that the univalent vertices are at the boundary of the disk.



- The web space consists of  $\mathbb{Z}[q, q^{-1}]$ -linear combinations of webs drawn in disks, modulo some relations

$$\langle \bigcirc \cup \Gamma \rangle = (q + q^{-1}) \langle \Gamma \rangle = \langle \begin{array}{c} \leftarrow \\ \rightarrow \\ \leftarrow \end{array} \cup \Gamma \rangle$$

$$\langle \begin{array}{c} \rightarrow \\ \leftarrow \\ \rightarrow \end{array} \rangle = \langle \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \rangle \quad \text{and} \quad \langle \begin{array}{c} \leftarrow \\ \rightarrow \\ \leftarrow \end{array} \rangle = \langle \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \rangle$$

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- By this relations, a closed web (a web with no boundary, thus no univalent vertices) with  $k$  connected components evaluates to  $(q + q^{-1})^k$ .
- The web space is a type of an oriented planar algebra.

# The $sl(2)$ quantum tangle (or link) invariant

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- Starting with a tangle diagram  $T$ , we evaluate each crossing by the rules:

$$\begin{array}{l} \begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \nearrow \end{array} = q \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} - q^2 \begin{array}{c} \nwarrow \\ \swarrow \\ \nwarrow \\ \swarrow \end{array} \\ \begin{array}{c} \nearrow \\ \swarrow \\ \nwarrow \\ \nearrow \end{array} = q^{-1} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} - q^{-2} \begin{array}{c} \nwarrow \\ \swarrow \\ \nwarrow \\ \swarrow \end{array} \end{array}$$

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- The quantum tangle invariant is a map of oriented planar algebras

**Oriented Tangles  $\rightarrow sl(2)$  web space**

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**Oriented Tangles  $\rightarrow sl(2)$  web space**

- If  $T$  is a link diagram  $D$ , its invariant is a Laurent polynomial in  $q$ ; in particular, it is the Jones polynomial associated to  $D$ , multiplied by  $q + q^{-1}$ .

# Categorification?

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To categorify the  $sl(2)$  quantum web space we want to associate (formal) *complexes* to tangles instead of linear combinations. Thus we want to replace

$$\begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \searrow \end{array} = q^{-1} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} - q^{-2} \begin{array}{c} \nwarrow \\ \swarrow \\ \nearrow \\ \nwarrow \end{array}$$

with some (formal) complex

$$\begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \searrow \end{array} = \left[ \begin{array}{c} 0 \longrightarrow q^{-1} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \longrightarrow ? \begin{array}{c} \nwarrow \\ \swarrow \\ \nearrow \\ \nwarrow \end{array} \longrightarrow 0 \end{array} \right]$$

## The category **Foams**( $B$ )

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$B$  is a finite set of points on a circle; in particular,  $B = \partial T$ .

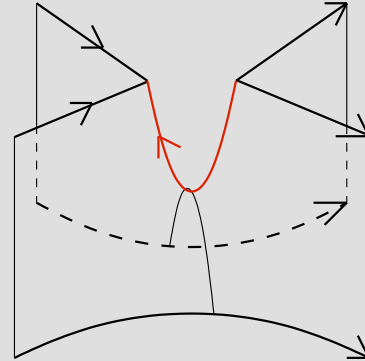
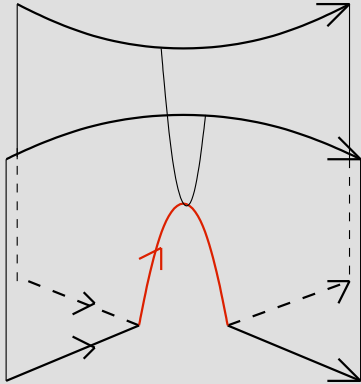
- objects: **web diagrams** with boundary  $B$  and *singular points* (the bivalent vertices) with neighborhoods homeomorphic to the letter  $V$ ;
- morphisms: cobordisms regarded up to boundary-preserving isotopies (called **foams**) whose tops and bottoms are webs and whose side boundaries are  $I \times B$ . Foams have *singular arcs* (or *singular circles*) where the orientations disagree, and near which they look like  $V \times [0, 1]$  (or  $V \times S^1$ );

# Foams' features

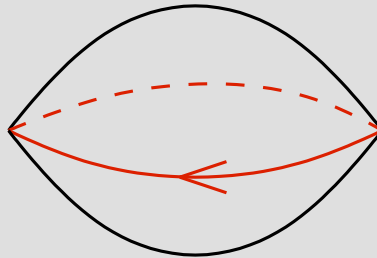
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- the singular arcs (or circles) have a preferred normal direction;
- two adjacent facets of a foam are compatibly oriented, inducing an orientation on singular arcs (or circles);
- we read foams (as morphisms) from bottom to top, by convention and we compose them by placing one atop the other;
- foams can have dots that are allowed to move freely along a facet they belong to, but can't cross singular arcs or circles. However, we can exchange dots between two adjacent facets, at an expense of a sign.

- **Example:** Some basic foams



- **Example:** closed foam



We call this closed foam the *ufo*-foam.

## Local relations

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Let  $a$  be a formal variable,  $i$  the primitive 4th root of unity, and consider the polynomial ring  $R = \mathbb{Z}[a, i]$ .

We mod out the morphisms of the category **Foams** by the local relations:

# Local relations

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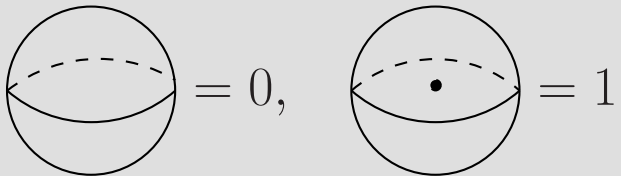
Let  $a$  be a formal variable,  $i$  the primitive 4th root of unity, and consider the polynomial ring  $R = \mathbb{Z}[a, i]$ .

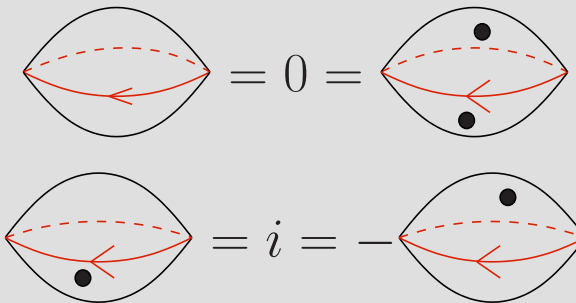
We mod out the morphisms of the category **Foams** by the local relations:

$$(2D) \quad \text{parallelogram with 2 dots} = a \text{ parallelogram}$$

$$(SF) \quad \text{cylinder} = \text{cup with dot} + \text{cup} + \text{cup} + \text{cup with dot}$$

## Closed foam relations

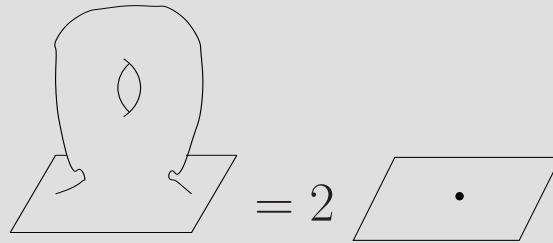
$$(S) \quad \text{Sphere} = 0, \quad \text{Sphere with dot} = 1$$


$$(UFO) \quad \text{Sphere with red arrow} = 0 = \text{Sphere with red arrow and two dots}$$
$$\text{Sphere with red arrow and one dot} = i = - \text{Sphere with red arrow and one dot}$$


We denote the new category by **Foams**<sub>/ℓ</sub>.

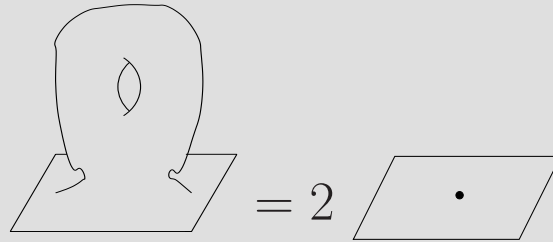
## Remark:

1. The surgery formula (SF) implies the genus reduction formula



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2. In particular we have that a torus evaluates to 2, a genus two or genus three, closed, connected and oriented surface evaluates to 0 and  $8a$ , respectively.



We introduce a grading on  $R = \mathbb{Z}[a, i]$  by letting

$$\deg(1) = 0 = \deg(i), \quad \deg(a) = 4.$$

**Definition.** If  $S$  is a foam (closed or not) with  $d$  dots in  $\mathbf{Foams}(B)$  we define the *grading* of  $S$  by  $\deg(S) = -\chi(S) + \frac{1}{2}|B| + 2d$ , where  $\chi$  is the Euler characteristic and  $|B|$  is the cardinality of  $B$ .

**Example:**

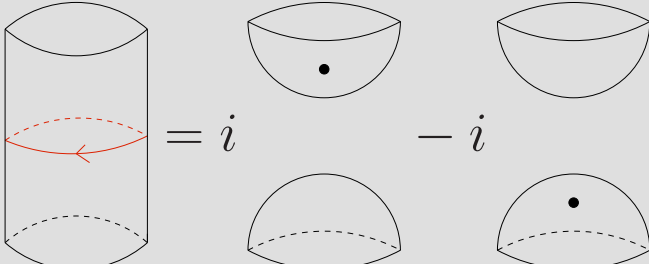
$$\begin{aligned} \deg \left( \text{cup with arrow} \right) &= \deg \left( \text{cup with arrow and red outline} \right) = -1 \\ \deg \left( \text{cup with arrow and dot} \right) &= \deg \left( \text{cup with arrow, red outline, and dot} \right) = 1 \end{aligned}$$

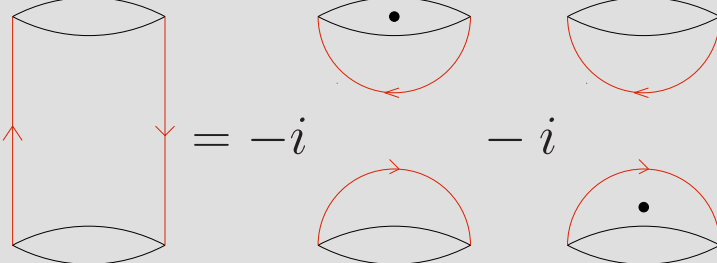
**Remark.** The category  $\mathbf{Foams}$  is graded and so is  $\mathbf{Foams}_{/\ell}$ , since the local relations are degree-preserving.

# Some useful relations

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The following relations hold in  $\mathbf{Foams}_{/e}$ :

(1)  (RSC)

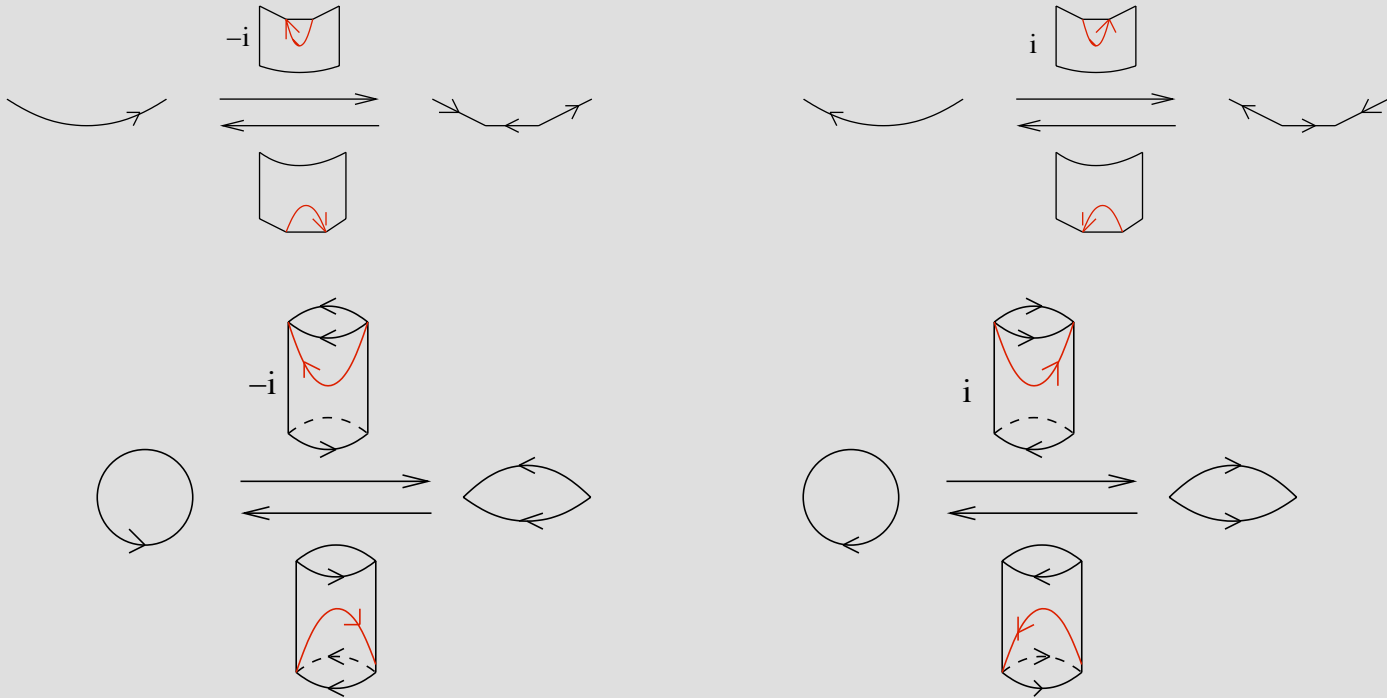
(2)  (CN)

where the dots in (CN) are on the back facets.

# Isomorphisms in $\mathbf{Foams}_{/\ell}$

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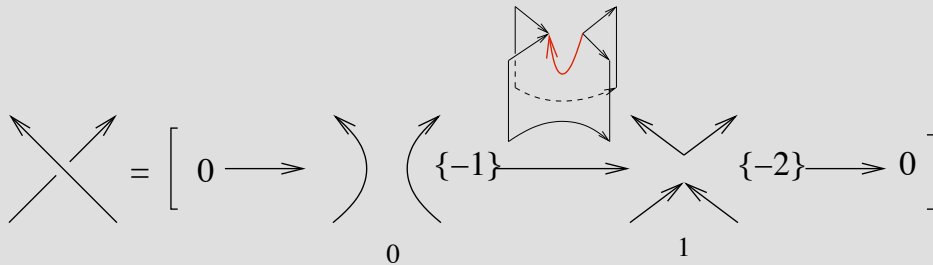
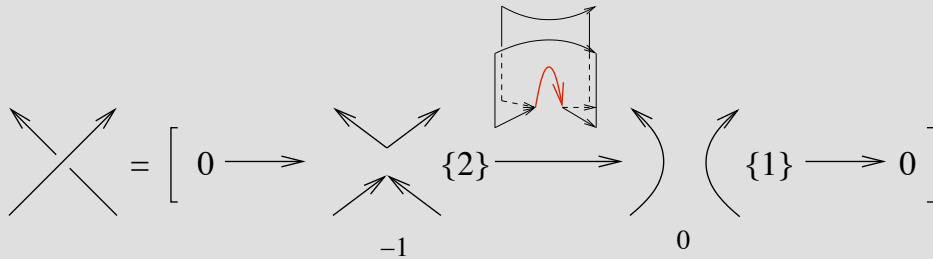
There are a few isomorphisms in the category  $\mathbf{Foams}_{/\ell}$  which allow for easy  $sl(2)$  knot homology computations:



# Chain complex associated to a tangle diagram

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Starting with a generic tangle diagram  $T$  with boundary points  $B$  we associate to it a chain complex  $[T]$  whose construction is explained by:



- The “chain objects” are “column vectors” of webs and “differentials” are “matrices of foams”.
- The complex  $[T]$  has differentials of degree 0.
- $[\cdot]$  is a map of oriented planar algebras

$$\mathbf{Oriented\ tangles} \longrightarrow \mathbf{Kom}_{/h}(\mathbf{Foams}_{/\ell})$$

and all planar algebra operations are of degree 0.

Here  $\mathbf{Kom}_{/h}(\mathbf{Foams}_{/\ell})$  is the category of complexes of (matrices of) foams modulo local relations, in which two homotopic morphisms are regarded the same.

# Invariance and Functoriality

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**Theorem.** The complex  $[T]$  is invariant under Reidemeister moves up to homotopy.

**Theorem.**  $[\cdot]$  is functorial under tangle cobordisms, i.e it induces a functor:

$$\mathbf{Cob}_{/i}^4 \longrightarrow \mathbf{Kom}_{/h}(\mathbf{Foams}_{/\ell})$$

where  $\mathbf{Cob}_{/i}^4$  is the category of four dimensional cobordisms between tangle diagrams.

# Applying a functor

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We are now ready to apply a specific functor  $\mathcal{F}$  to pass to an algebraic category

$$\mathbf{Foams}_{/\ell} \xrightarrow{\mathcal{F}} R - \mathbf{Mod}$$

- By doing this, we obtain an ordinary chain complex which is, up-to-homotopy, an invariant of  $T$ .
- $\mathcal{F}(\emptyset) = R = \mathbb{Z}[a, i]$ .

**Theorem.** The isomorphism class of the homology  $H(\mathcal{F}([T]))$  is a bigraded invariant of  $T$ .

## How is the functor $\mathcal{F}$ defined?

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Consider the category  $R\text{-Mod}$  of  $R$ -modules and module homomorphisms. Let  $\Gamma_0$  be a web in  $\mathbf{Foams}_{/\ell}(B)$ .

Define a functor  $\mathcal{F}_{\Gamma_0} : \mathbf{Foams}_{/\ell}(B) \longrightarrow R\text{-Mod}$  as follows:

- on objects:

$$\mathcal{F}_{\Gamma_0}(\Gamma) := \text{Mor}_{\mathbf{Foam}_{/\ell}(B)}(\Gamma_0, \Gamma)$$

- on morphisms: if  $S : \Gamma' \rightarrow \Gamma''$  then

$$\mathcal{F}_{\Gamma_0}(S) : \text{Mor}(\Gamma_0, \Gamma') \rightarrow \text{Mor}(\Gamma_0, \Gamma'')$$

maps  $U \in \text{Mor}(\Gamma_0, \Gamma')$  to  $S \circ U \in \text{Mor}(\Gamma_0, \Gamma'')$ .

If  $B = \emptyset$ , we consider  $\Gamma_0 = \emptyset$ , the empty web.

# $\mathcal{F}(\Gamma)$ mimics the $sl(2)$ quantum web space skein relations

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Consider  $\mathcal{A} = \mathbb{Z}[a, i, X]/(X^2 - a)$ , the rank two  $\mathbb{Z}[a, i]$ -module, with generators 1 and  $X$ , having degrees  $-1$  and  $1$ , respectively.

**Proposition:** There are canonical isomorphisms of graded abelian groups:

$$1. \mathcal{F}(\Gamma \cup \bigcirc) \cong \mathcal{F}(\Gamma) \otimes_{\mathbb{Z}[a, i]} \mathcal{A}$$

$$2. \mathcal{F}(\Gamma \cup \begin{array}{c} \leftarrow \\ \rightarrow \end{array}) \cong \mathcal{F}(\Gamma) \otimes_{\mathbb{Z}[a, i]} \mathcal{A}$$

$$3. \mathcal{F}(\begin{array}{c} \searrow \\ \leftarrow \\ \nearrow \end{array}) \cong \mathcal{F}(\begin{array}{c} \curvearrowright \end{array})$$

$$4. \mathcal{F}(\begin{array}{c} \searrow \\ \rightarrow \\ \nearrow \end{array}) \cong \mathcal{F}(\begin{array}{c} \curvearrowleft \end{array})$$

## The case of links

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- The ‘homology’ of each resolution is isomorphic to  $\mathcal{A}^{\otimes k}$ , where  $k$  is the number of connected components of that resolution.
- In particular,  $\mathcal{F}(\Gamma)$  is a free  $\mathbb{Z}[a, i]$ -module of graded rank  $\langle \Gamma \rangle$ .
- The invariant of a link is a complex of graded free modules, whose graded Euler characteristic is equal to the  $sl(2)$  quantum link invariant.

# A Frobenius algebra and a TQFT with dots

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When restricted to links, the functor  $\mathcal{F}$  can be regarded as a TQFT with dots, where a dot stands for multiplication by  $X$  endomorphism of the algebra  $\mathcal{A}$ .

More precisely, our link invariant is the homology theory assigned to the Frobenius system  $(\mathbb{Z}[a, i], \mathcal{A}, \epsilon, \Delta)$ , where

$$\iota : \mathbb{Z}[a, i] \rightarrow \mathcal{A}, \quad \iota(1) = 1,$$

$$\epsilon : \mathcal{A} \rightarrow \mathbb{Z}[a, i], \quad \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

$$\epsilon(1) = 0, \quad \Delta(1) = 1 \otimes X + X \otimes 1$$

$$\epsilon(X) = 1, \quad \Delta(X) = X \otimes X + a1 \otimes 1,$$

$$m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad \begin{cases} m(1, X) = X & m(1, 1) = 1 \\ m(X, 1) = X & m(X, X) = a. \end{cases}$$

# Conclusions

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- For  $a = 0$ , this construction is isomorphic to the original Khovanov homology theory, and being properly functorial with respect to link cobordisms, it naturally resolves the sign ambiguity in functoriality of the mentioned theory.
- For  $a = 1$ , this is the Lee variant of the original Khovanov theory.
- Since a 2-knot in  $\mathbb{R}^4$  is a cobordism from the empty link to itself, we obtain an invariant of 2-knots.