On Khovanov-Rozansky cohomology

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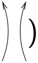
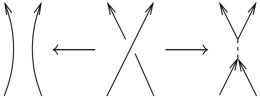
Plan

- We construct a bigraded link cohomology theory that depends on two parameters h and a, and which categorifies the $\mathfrak{sl}(2)$-quantum polynomial.
- This theory is the "$\mathfrak{sl}(2)$-equivariant" version of the Khovanov-Rozansky $\mathfrak{sl}(2)$-link cohomology .

$\mathfrak{sl}(2)$-quantum polynomial invariant P_2

- $\{\textit{oriented links in } S^3\} \xrightarrow{P_2} \mathbb{Z}[q, q^{-1}]$
- $P_2(L_1 \cup L_2) = P_2(L_1)P_2(L_2)$
- $q^2 P_2($  $) - q^{-2} P_2($  $) = (q - q^{-1}) P_2($  $)$
- $P_2($  $) = q + q^{-1}$

Resolving crossings

- Resolve a crossing in two ways: 

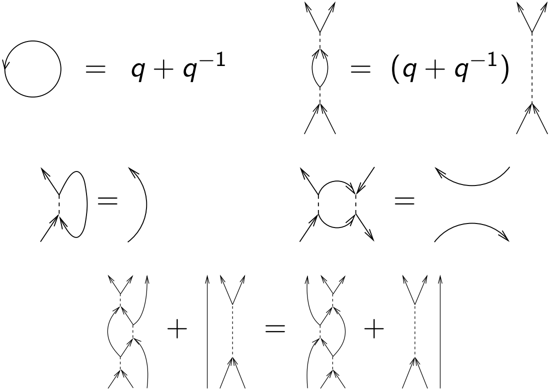
- n-crossing link $L \rightarrow 2^n$ resolutions Γ

- Γ: planar graph with bivalent vertices of type:  or 

- $\Gamma \longrightarrow$ the graph polynomial $P_\Gamma(q) \in \mathbb{Z}[q, q^{-1}]$

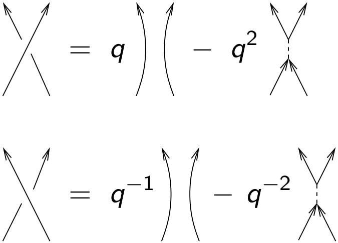
The graph polynomial $P_\Gamma(q)$

Compute $P_\Gamma(q)$ so that is satisfies the skein relations:



Computing the polynomial invariant $P_2(L)$

Compute $P_2(L)$ by:



$$P_2(L) = \sum_{\textit{resolutions}\ \Gamma} \pm q^{\alpha(\Gamma)} P_\Gamma(q), \quad \text{where } \alpha(\Gamma) \in \mathbb{Z}$$

To categorify $P_2(L)$, we first categorify $P_\Gamma(q)$:

- $\Gamma \longrightarrow \overline{C}(\Gamma)$: $\quad (\mathbb{Z} \oplus \mathbb{Z}_2)$-graded $\mathbb{Q}[a, h]$-module
- a and h are formal variables and $\mathbb{Q}[a, h]$ is graded by:

$$\deg a = 4 \quad \text{and} \quad \deg h = 2$$

- The construction of $\overline{C}(\Gamma)$: via matrix factorization

Matrix factorizations

Definition. A matrix factorization M over a commutative ring R consists of two free R-modules and two R-module maps

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0$$

such that $d^2 = \omega$, where $\omega \in R$ is called potential.
We write $M \in MF_\omega$.

Matrix factorizations for closed graphs

If Γ is a closed graph (has no boundary points):

- $\Gamma \longrightarrow \overline{C}(\Gamma) \in MF_{\omega=0}$
- $\overline{C}(\Gamma)$ is a 2-periodic complex of graded $\mathbb{Q}[a, h]$-modules
- Thus we can take $H(\overline{C}(\Gamma))$

Construction of $\overline{C}(\Gamma)$

- $\underset{\rule{2cm}{0.4pt}}{e} \longrightarrow \quad \omega(x_e): \; = p(a,h,x_e) \in \mathbb{Q}[a,h,x_e]$

  so that $\frac{1}{3}\,\partial_{x_e} p = x_e^2 - hx_e - a$

  so that $\deg x_e = 2, \; \deg \omega(x_e) = 6$

Construction of $\overline{C}(\Gamma)$

- $\underset{\rule{3cm}{0.4pt}}{e} \longrightarrow \quad \omega(x_e): \; = p(a, h, x_e) \in \mathbb{Q}[a, h, x_e]$

  so that $\frac{1}{3}\,\partial_{x_e} p = x_e^2 - hx_e - a$

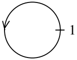
  so that $\deg x_e = 2, \; \deg \omega(x_e) = 6$

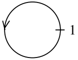
- $1 \longrightarrow 2 \quad \longrightarrow \quad \overline{C}(1 \longrightarrow 2) \in MF_{\omega(x_2)-\omega(x_1)}$

  write $\omega(x_2) - \omega(x_1) = (x_2 - x_1)p_{12}$

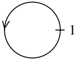
- $\overline{C}(1 \longrightarrow 2): \quad R \xrightarrow{p_{12}} R\{-1\} \xrightarrow{x_2 - x_1} R, \; R = \mathbb{Q}[a, h, x_1, x_2]$
- We call it a short factorization

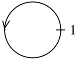
Construction of $\overline{C}(\Gamma)$

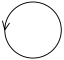
- $\overline{C}($  $) = \overline{C}(1 \longrightarrow 2)_{/(x_2 = x_1)} \in MF_{\omega=0}$

- We can compute the homology of $\overline{C}($  $)$

Construction of $\overline{C}(\Gamma)$

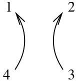
- $\overline{C}($  $) = \overline{C}(1 \longrightarrow 2)_{/(x_2 = x_1)} \in MF_{\omega=0}$

- We can compute the homology of $\overline{C}($  $)$

- $H(\overline{C}($  $)) = \mathbb{Q}[a,h,x]/_{(x^2 - hx - a)}\{-1\}$

- $q\,\mathrm{rk}_{\mathbb{Q}[a,h]}\, H(unknot) = q + q^{-1} = P$  (q)

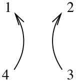
Construction of $\overline{C}(\Gamma)$

$\overline{C}($  $) := \overline{C}($  $) \otimes \overline{C}($  $) \in MF_{\omega(x_1)+\omega(x_2)-\omega(x_3)-\omega(x_4)}$

$$= \left(R \xrightarrow{p_{41}} R\{-1\} \xrightarrow{x_1 - x_4} R\right) \otimes$$

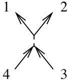
$$\otimes \left(R \xrightarrow{p_{32}} R\{-1\} \xrightarrow{x_2 - x_3} R\right)$$

where $R = \mathbb{Q}[a, h, x_1, x_2, x_3, x_4]$

$\omega($  $) = \omega(x_1) + \omega(x_2) - \omega(x_3) - \omega(x_4)$

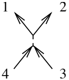
$$= (x_1 - x_4)p_{41} + (x_2 - x_3)p_{32}$$

Construction of $\overline{C}(\Gamma)$

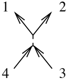
$\overline{C}($ 

$)$: $$= \left(R \xrightarrow{a_1} R\{-1\} \xrightarrow{x_1+x_2-x_3-x_4} R \right) \otimes$$
$$\otimes \left(R\{-1\} \xrightarrow{a_2} R \xrightarrow{x_1x_2-x_3x_4} R\{-1\} \right)$$

for some $a_1, a_2 \in R = \mathbb{Q}[a, h, x_1, x_2, x_3, x_4]$

$\overline{C}($ 

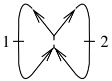
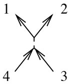
$) \in MF_{\omega(x_1)+\omega(x_2)-\omega(x_3)-\omega(x_4)}$

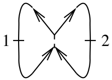
$\omega($ 

$) = \omega(x_1) + \omega(x_2) - \omega(x_3) - \omega(x_4)$

$$= (x_1 + x_2 - x_3 - x_4)a_1 + (x_1x_2 - x_3x_4)a_2$$

Construction of $\overline{C}(\Gamma)$

- $\overline{C}($  $):= \overline{C}($  $)/_{(x_4=x_1, x_3=x_2)} \in MF_{\omega=0}$

- $H(\overline{C}($  $)) = \mathbb{Q}[a,h,x_1,x_2]/_{(x_1+x_2-h,\, x_1x_2+a)}\{-1\}$

  $\cong \mathbb{Q}[a,h,x]/_{(x^2-hx-a)}\{-1\}$

Construction of $\overline{C}(\Gamma)$

- closed graph $\Gamma \to \overline{C}(\Gamma)$: the tensor product over all singular resolutions of the form  and over all oriented loops
- $\omega(\overline{C}(\Gamma)) = 0$
- $\overline{C}(\Gamma)$ has homology in one degree only, and we define

$$H(\Gamma) := H(\overline{C}(\Gamma)), \quad H(\Gamma) = \oplus_j H^j(\Gamma)$$

Construction of $\overline{C}(\Gamma)$

Proposition

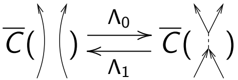
Factorizations $\overline{C}(\Gamma)$ mimic the skein relations used to define $P_\Gamma(q)$.

Theorem

The graded rank of the homology of Γ is the graph polynomial $P_\Gamma(q)$:

$$P_\Gamma(q) = \sum_j q^j \operatorname{rk}_{\mathbb{Q}[a,h]} H^j(\Gamma) = q \operatorname{rk}_{\mathbb{Q}[a,h]} H(\Gamma).$$

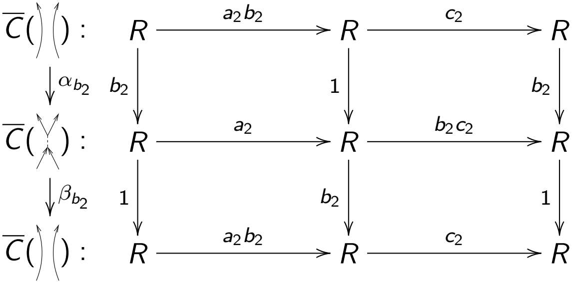
Homomorphisms of matrix factorizations



$$\begin{array}{ccc} R \xrightarrow{a_1} R \xrightarrow{b_1} R & & R \xrightarrow{a_1} R \xrightarrow{b_1} R \\ \otimes & \rightleftarrows & \otimes \\ R \xrightarrow{a_2 b_2} R \xrightarrow{c_2} R & & R \xrightarrow{a_2} R \xrightarrow{b_2 c_2} R \end{array}$$

$$\Lambda_0 = \mathrm{Id} \otimes \alpha_{b_2} \quad \text{and} \quad \Lambda_1 = \mathrm{Id} \otimes \beta_{b_2}$$

Homomorphisms of matrix factorizations



A functor

Definition

Denote by *Foams* the category whose objects are planar bivalent graphs Γ and whose morphisms are singular cobordisms (*foams*) between such graphs.
Denote by hmf_ω the homotopy category of graded matrix factorizations with potential ω.

Definition

We denote by $\overline{C}\colon\ Foams \to hmf_\omega$ the functor that associates to a graph Γ the factorization $\overline{C}(\Gamma)$ and to a foam a homomorphism of factorizations.

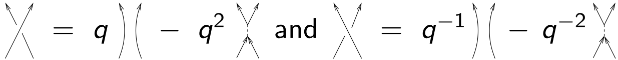
Categorifying the polynomial invariant $P_2(L)$

link diagram $L \longrightarrow C(L)$ complex of bigraded $\mathbb{Q}[a, h]$-modules such that:

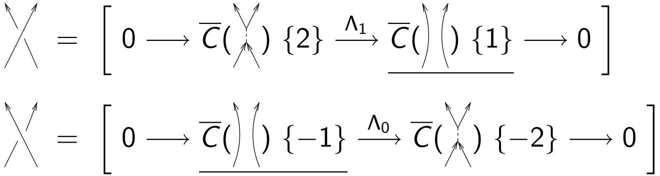
1. $L_1 \sim L_2$ (Reidemeister move) $\Rightarrow C(L_1) \overset{homotopy}{\cong} C(L_2)$
2. $\chi_q(C(L)) = P_2(L)$

Categorifying the polynomial invariant $P_2(L)$

- crossing $p \longrightarrow C_p$ = complex of factorizations
- Replace



by:



- L link diagram $\longrightarrow C(L) = \otimes C_p$ over all crossings in L
- $C(L)$ is a complex of graded $\mathbb{Q}[a, h]$-modules

Categorifying the polynomial invariant $P_2(L)$

Theorem

$C(L)$ is invariant under the Reidemeister moves, up to homotopy. Thus, the isomorphism class of $C(L)$ in the category $Kom(hmf_\omega)$ is an invariant of L.

$$\mathcal{H}(L) := \oplus_{i,j \in \mathbb{Z}} H^{i,j}(C(L))$$

Proposition

$$\chi(\mathcal{H}(L)) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \, \mathrm{rk}_{\mathbb{Q}[a,h]} H^{i,j}(C(L)) = P_2(L)$$

A TQFT functor?

Consider again the functor $\overline{C}\colon\ \textit{Foams} \to \textit{hmf}_{\omega}$ and the ordinary homology functor $\mathcal{H}$. Then we have

$$\mathcal{H} \circ \overline{C}\colon\ \textit{Foams} \to \mathbb{Q}[a, h] - \mathrm{Mod}$$

where $\mathbb{Q}[a, h] - \mathrm{Mod}$ is the category of $\mathbb{Q}[a, h]$-modules and module homomorphisms.

Theorem

The functor $\mathcal{H} \circ \overline{C}$ behaves similarly to a certain $(1 + 1)$-dimensional TQFT functor $\mathcal{F}\colon\ \textit{OrCob} \to \mathbb{Q}[a, h]\text{-}\textit{Mod}$.

A TQFT functor?

$\mathcal{F} \Longleftrightarrow$ Frobenius system $(\mathbb{Q}[a,h], \mathcal{A}, \epsilon, \Delta)$

- $\mathcal{A} = \mathbb{Q}[a,h,X]/(X^2 - hX - a) = < 1, X >_{\mathbb{Q}[a,h]}$
  $\deg 1 = -1, \ \deg X = 1$
- $\epsilon \colon \ \mathcal{A} \to \mathbb{Q}[a,h], \epsilon(1) = 0, \epsilon(X) = 1$
- $m \colon \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \begin{cases} m(1,X) = X & m(1,1) = 1 \\ m(X,1) = X & m(X,X) = hX + a \end{cases}$
- $\Delta \colon \ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \begin{cases} \Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ \Delta(X) = X \otimes X + a1 \otimes 1 \end{cases}$

THANK YOU!