Link cohomology and extended TQFTs

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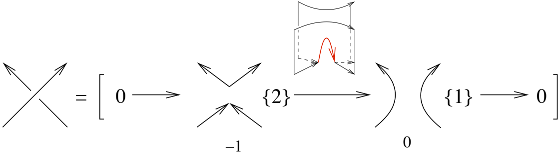
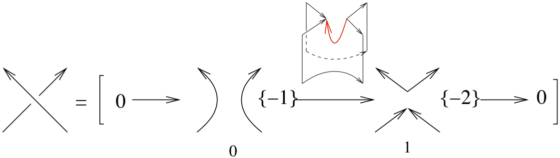
Outline

- Review of the universal $\mathfrak{sl}(2)$ foam cohomology with no dots
- Some definitions
- $\mathfrak{sl}(2)$ foam link cohomology via generalized 2D TQFTs

Constructing complexes from link diagrams

- Let D be a diagram of an oriented link L.
- $D \longrightarrow [D]$, a *formal complex* obtained by taking the tensor product (over all crossings in D) of chains of the form:



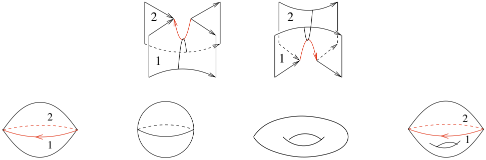


Constructing complexes from link diagrams

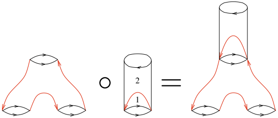
- The "chain objects" of $[D]$ are column vectors of *webs* and "differentials" are matrices of *foams*.
- Webs are disjoint unions of piecewise oriented loops containing bivalent vertices that look like  and .
- One of the two edges that share a bivalent vertex is the *preferred edge* for the corresponding vertex.
- A foam is a cobordism between two webs, regarded up to boundary-preserving isotopies; more precisely, a foam is a piecewise oriented 2-cobordism with *singular arcs/circles* where orientations disagree.
- The singular arcs/circles have a *preferred neighboring facet*.

Some examples of foams

- The two facets on the two sides of a singular arc/circle are oppositely oriented, inducing an orientation on the singular arc/circle.



- We read foams as morphisms from top to bottom, by convention, and we compose them by placing one on top the other:

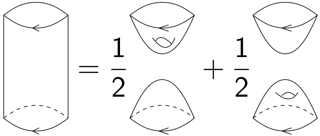


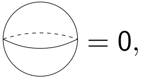
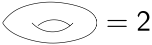
Local relations

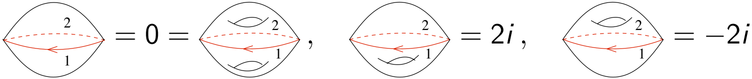
- If S is a foam we define the *degree* of S by $\deg(S) = -\chi(S)$.
- The formal complex $[D]$ is not a link invariant. However:
- We introduce a set of relations ℓ among foams.
  Denote by $\mathbf{Foams}_{/\ell}$ the category whose objects are webs and whose morphisms are foams regarded modulo these relations.
- The formal complex $[D]$ is invariant under Reidemester moves, up to homotopy, when regarded as an object of the category of complexes over $\mathbf{Foams}_{/\ell}$.

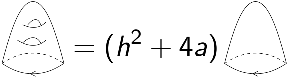
Local relations

Let a and h be formal parameters with $\deg(a) = 4$ and $\deg(h) = 2$.

(SF) 

(S)  $= 0$, (T)  $= 2$

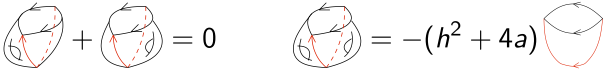
(UFO) 

(G2) 

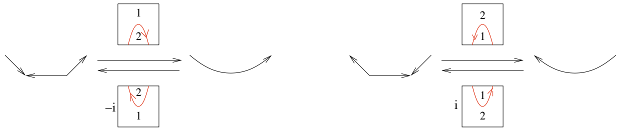
Some identities implied by ℓ

The following identities are implied by the local relations ℓ:

- rules for *exchanging handles* between neighboring facets:



- isomorphisms in the category $\mathbf{Foam}_{/\ell}$:



Passing to an algebraic category

- We apply a degree-preserving ‘tautological’ functor

$$\mathbf{Foams}_{/\ell} \xrightarrow{\mathcal{F}} R - \mathbf{Mod}$$

where $R := \mathbb{Z}[\frac{1}{2}, i][a, h]$ is graded by: $\deg(a) = 4, \deg(h) = 2$.

- This functor is defined on objects by $\mathcal{F}(\Gamma) := \mathrm{Mor}_{\mathbf{Foam}_{/\ell}}(\emptyset, \Gamma)$ and on morphisms by composition on the left.

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- The isomorphism class of the homology of the resulting complex $\mathcal{H}(L) := H(\mathcal{F}([D]))$ is a bi-graded invariant of L, whose graded Euler characteristic is the $\mathfrak{sl}(2)$ polynomial of the link L with diagram D.
- The theory is *properly functorial* with respect to link cobordisms (rel. boundary).

- *Goal:* to obtain the universal $\mathfrak{sl}(2)$ cohomology theory for links using some type of TQFT.
- What maps should be associated to singular arcs/circles?
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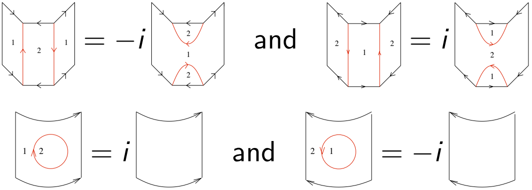
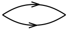
- Consider again the category **Foams** and impose the following *curtain identities*:



- Then each web is isomorphic to , called *bi-web*.

The category **eSing-2Cob**

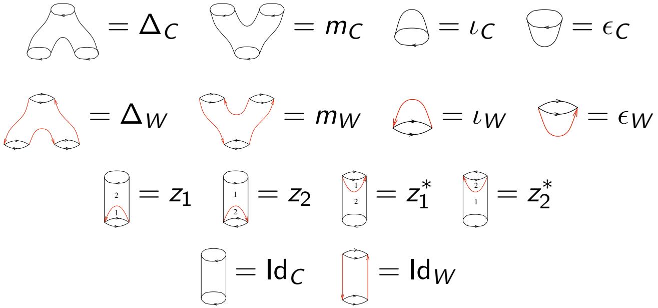
Therefore, we can talk about a new category, **eSing-2Cob**

- whose objects are finite sequences of bi-webs, and clockwise and counterclockwise oriented circles:

- and whose morphisms are diffeomorphism classes of piecewise oriented cobordisms between them (called *extended singular 2-cobordisms*).

Generators of the category **eSing-2Cob**

Every (connected) morphism in **eSing-2Cob** can be obtained by gluing the following singular cobordisms:



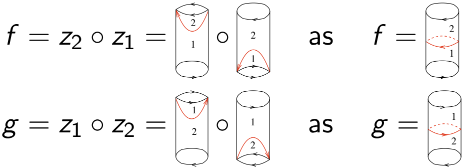
$= \Delta_C$ $= m_C$ $= \iota_C$ $= \epsilon_C$

$= \Delta_W$ $= m_W$ $= \iota_W$ $= \epsilon_W$

$= z_1$ $= z_2$ $= z_1^*$ $= z_2^*$

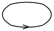
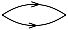
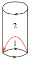
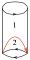
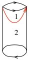
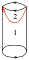
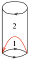
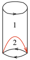
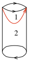
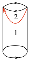
$= \mathrm{Id}_C$ $= \mathrm{Id}_W$

We draw the composite cobordisms



Relations in **eSing-2Cob**

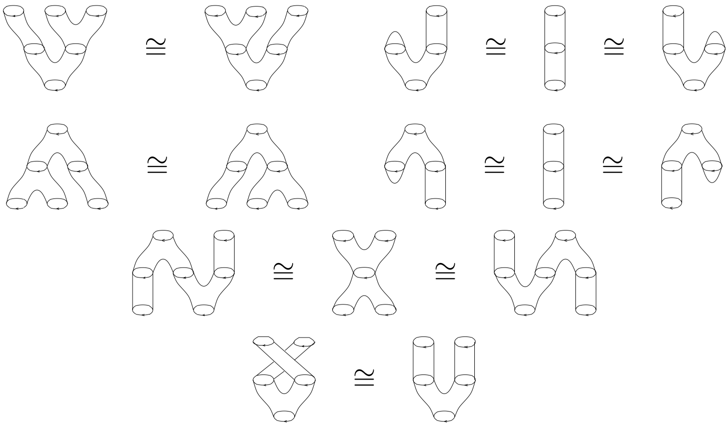
Relations among the generating cobordisms in **eSing-2Cob** :

-  and  form commutative Frobenius algebra objects.
-  forms a symmetric Frobenius algebra object.
- The "zipper" cobordisms ,  are algebra homomorphisms.
- The "cozipper" cobordisms ,  are dual to the zippers.
- The image of a circle under each zipper is a central element.
- The zipper  (respectively ) is an isomorphism with inverse i  (respectively $-i$ ).

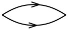
These relations are the axioms defining an *extended twin Frobenius algebra*.

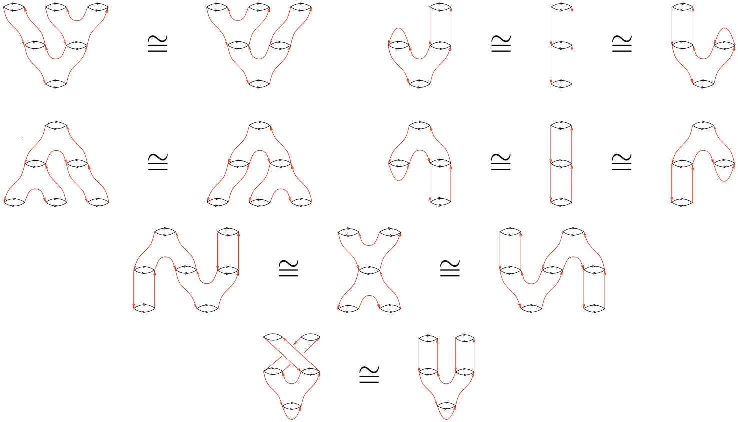
Relations

1. The circle forms a commutative Frobenius algebra object.

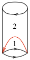


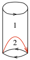
Relations

2. The bi-web  forms a symmetric Frobenius algebra object.

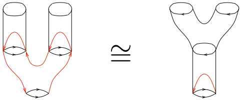


Relations

3. The "zipper" 

(and 

) forms an algebra homomorphism.



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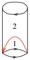
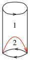
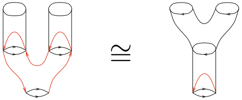
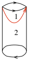
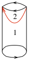
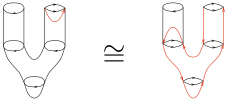
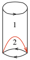
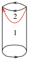
3. The "zipper"  (and ) forms an algebra homomorphism.

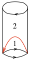


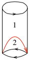
4. The "cozipper"  (and ) is dual to the zipper.



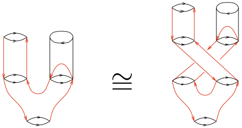
There are similar relations for  and  .

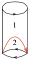
Relations

5. The image of the circle under the zipper 

(and 

) is a central element.



There is a similar relation for 

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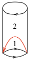
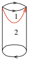
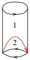
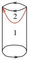
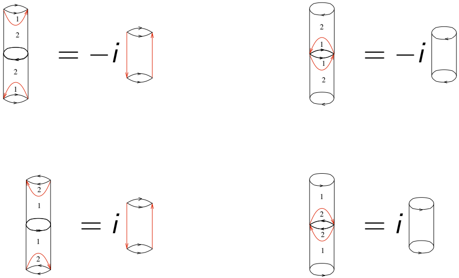
Relations

6.  and i  (as well as  and $-i$ ) are mutually inverse isomorphisms.



Definition

An *extended twin Frobenius algebra* $\mathbf{T} := (C, W, z_1, z_1^*, z_2, z_2^*)$ consists of

- a commutative Frobenius algebra $C = (C, m_C, \iota_C, \Delta_C, \epsilon_C)$
- a symmetric Frobenius algebra $W = (W, m_W, \iota_W, \Delta_W, \epsilon_W)$
- four morphisms $z_{1,2} \colon C \to W$ and $z_{1,2}^* \colon W \to C$ of $\mathcal{C}$

such that z_1 and z_2 are algebra homomorphisms satisfying

1. z_i^* is the morphism dual to z_i, for $i = 1, 2$
2. the image of C under z_i $(i = 1, 2)$ is contained in the center of W
3. z_1 and iz_1^*, as well as z_2 and $-iz_2^*$ are mutually inverse isomorphisms.

Proposition

The category **eSing-2Cob** *is the strict symmetric monoidal category freely generated by an extended twin Frobenius algebra.*

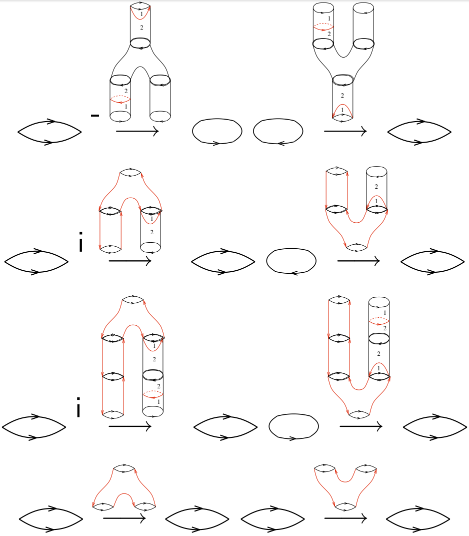
An extended twin Frobenius algebra defines a 2-dimensional extended twin TQFT.

Definition

An *extended twin Topological Quantum Field Theory* is a symmetric monoidal functor

$$\mathcal{T} : \textbf{eSing-2Cob} \to \textbf{R-Mod}.$$

- Given an oriented link diagram, we consider the associated hypercube of resolutions as in the foam theory, with the same degree shift of each vertex as before.
- Whenever a web has more than two vertices, we apply the curtain identities and replace that web by the bi-web.
- Oriented circles are let unchanged.
- The vertices of the resulting hypercube $\mathcal{U}$ are objects in **eSing-2Cob**.
- The morphism decorating an edge of the resulting hypercube $\mathcal{U}$ is a disjoint union of cylinders over an oriented circle and/or bi-web with a saddle cobordism that looks like one of the following:



up to permutation of the two 1-manifolds in the source/target of the singular saddle cobordism.

- These singular saddle cobordisms are compositions of generating morphisms of the category **eSing-2Cob**.
- Vertices and edges of the hypercube $\mathcal{U}$ are decorated by objects and morphisms in **eSing-2Cob**, respectively.
- Each square face of this hypercube commutes; we sprinkle some edges with minus signs, so that each square face of $\mathcal{U}$ anti-commutes.
- We apply a graded *extended twin TQFT*

$$\mathcal{T} : \textbf{eSing-2Cob} \to \textbf{R-Mod}$$

  and turn $\mathcal{U}$ into an anti-commutative cube $\mathcal{T}(\mathcal{U})$ in **R-Mod**.
- The degree shift of each vertex assures that the edges of the hypercubes $\mathcal{U}$ and $\mathcal{T}(\mathcal{U})$ are graded maps of degree zero.

Extended twin Frobenius algebra - example

- Here is the extended twin Frobenius algebra that we are concerned with.
- Consider the graded R-module $\mathcal{A} = R[X]/(X^2 - hX - a)$ where $R := \mathbb{Z}[\frac{1}{2}, i][a, h]$, with $\deg(1) = -1$ and $\deg(X) = 1$, equipped with two commutative Frobenius structures (differing by a twist):

$$\mathcal{A}_C = (\mathcal{A}, m_C, \iota_C, \Delta_C, \epsilon_C), \quad \mathcal{A}_W = (\mathcal{A}, m_W, \iota_W, \Delta_W, \epsilon_W),$$

$$\begin{cases} \Delta_C(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ \Delta_C(X) = X \otimes X + a1 \otimes 1 \end{cases} \qquad \begin{cases} \epsilon_C(1) = 0 \\ \epsilon_C(X) = 1 \end{cases}$$

$$\begin{cases} \Delta_W(1) = i(1 \otimes X + X \otimes 1 - h1 \otimes 1) \\ \Delta_W(X) = i(X \otimes X + a1 \otimes 1) \end{cases} \qquad \begin{cases} \epsilon_W(1) = 0 \\ \epsilon_W(X) = -i. \end{cases}$$

Extended twin Frobenius algebra - example

- We define the following isomorphisms:

$$z_1\colon \mathcal{A}_C \to \mathcal{A}_W, \begin{cases} 1 \mapsto 1 \\ X \mapsto X, \end{cases} \quad z_1^*\colon \mathcal{A}_W \to \mathcal{A}_C, \begin{cases} 1 \mapsto -i \\ X \mapsto -iX, \end{cases}$$

$$z_2\colon \mathcal{A}_C \to \mathcal{A}_W, \begin{cases} 1 \mapsto 1 \\ X \mapsto h - X, \end{cases} \quad z_2^*\colon \mathcal{A}_W \to \mathcal{A}_C, \begin{cases} 1 \mapsto i \\ X \mapsto i(h - X). \end{cases}$$

- $(\mathcal{A}_C, \mathcal{A}_W, z_1, z_1^*, z_2, z_2^*)$ satisfies the axioms of an extended twin Frobenius algebra.
- $\mathcal{A}_W$ is a "twist" of $\mathcal{A}_C$ by the invertible element $-i$:

$$\epsilon_W(g) = \epsilon_C(-ig), \ \Delta_W(g) = \Delta_C(ig), \quad \forall g \in \mathcal{A}.$$

Extended twin TQFT - example

The extended twin Frobenius algebra $(\mathcal{A}_C, \mathcal{A}_W, z_1, z_1^*, z_2, z_2^*)$ gives rise to an extended twin TQFT

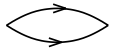
$$\mathcal{T} : \textbf{eSing-2Cob} \to \textbf{R-Mod}$$

On objects, $\mathcal{T}$ is defined as follows:

- $\mathcal{T}(\text{empty 1-manifold}) = R$
- $\mathcal{T}(\mathbf{n}) = \mathcal{A}_i^{\otimes k}$, for $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ a k-component object, where

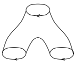
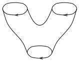
$\mathcal{A}_i = \mathcal{A}_C$ for $n_i =$ ,

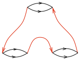
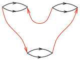
$\mathcal{A}_i = \mathcal{A}_C$ for $n_i =$ ,

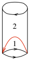
$\mathcal{A}_i = \mathcal{A}_W$ for $n_i =$ .

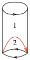
Extended twin TQFT - example

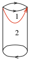
On the generating morphisms, $\mathcal{T}$ is defined as follows:

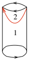
$\mathcal{T}:$  $\to \Delta_C$ $\mathcal{T}:$  $\to m_C$ $\mathcal{T}:$  $\to \iota_C$ $\mathcal{T}:$  $\to \epsilon_C$

$\mathcal{T}:$  $\to \Delta_W$ $\mathcal{T}:$  $\to m_W$ $\mathcal{T}:$  $\to \iota_W$ $\mathcal{T}:$  $\to \epsilon_W$

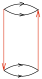
$\mathcal{T}:$ 

$\to z_1$ $\mathcal{T}:$ 

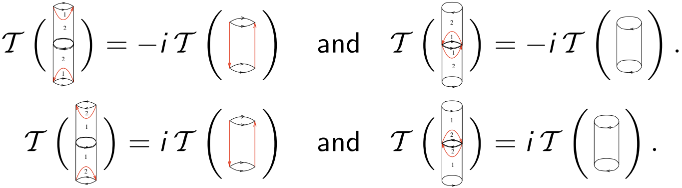
$\to z_2$ $\mathcal{T}:$ 

$\to z_1^*$ $\mathcal{T}:$ 

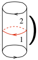
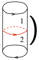
$\to z_2^*$

$\mathcal{T}:$  $\to \mathrm{Id}_{\mathcal{A}_C}$ $\mathcal{T}:$  $\to \mathrm{Id}_{\mathcal{A}_W}$.

Properties of this TQFT



Therefore, $\mathcal{T}$ satisfies the curtain identities we have imposed on the set of morphisms in **eSing-2Cob**.

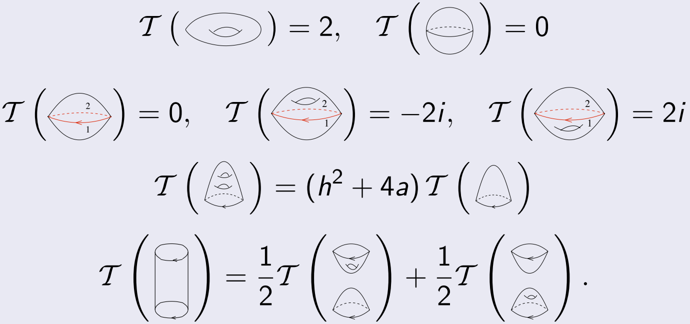
Also $\mathcal{T}($$) : \begin{cases} 1 \to i \\ X \to i(h-X) \end{cases}$ and $\mathcal{T}($$) : \begin{cases} 1 \to -i \\ X \to -i(h-X). \end{cases}$

$\mathcal{T}($$) = \mathcal{A}_C \cong <1, X>_R$ and $\mathcal{T}($$) = \mathcal{A}_C \cong <1, h-X>_R$.

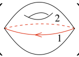
Properties of this TQFT

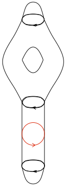
Proposition

The functor $\mathcal{T}$ satisfies the local relations ℓ of the universal $\mathfrak{sl}(2)$ foam cohomology, that is,



Proof: It's a routine. For example, to show that

$\mathcal{T}\Big($ 

$\Big) = \mathcal{T}\Big($  $\Big) = -2i,$

we compute

$$\begin{aligned}(\epsilon_C \circ z_1^* \circ z_1 \circ m_C \circ \Delta_C \circ \iota_C)(1) &= \\ (\epsilon_C \circ z_1^* \circ z_1)(2X - h) &= \\ -2i\,\epsilon(X) + ih\,\epsilon(1) &= -2i.\end{aligned}$$

The link invariant

- We have obtained an anti-commutative hypercube $\mathcal{T}(U)$ in **R-Mod** associated to a link diagram D.
- We form a chain complex $\mathcal{C}(D)$ using this hypercube, in the usual fashion.
- $\mathcal{C}(D)$ is an object in the category Kom(**R-Mod**).

Theorem

If D and D' are oriented link diagrams that are related by a Reidemeister move, then the complexes $\mathcal{C}(D)$ are $\mathcal{C}(D')$ are homotopy equivalent.

Proof: via 'delooping' and 'Gaussian elimination' (for chain complexes) tools.

Lemma

The following isomorphisms hold in **R-Mod***:*

$\mathcal{T}($  $) \cong R\{1\} \oplus R\{-1\},$

$\mathcal{T}($  $) \cong R\{1\} \oplus R\{-1\}.$

Proof:

$\alpha_C = \Big(\mathcal{T}($  $), \mathcal{T}(\frac{1}{2}$  $+ \frac{h}{2}$  $)\Big)^t, \ \beta_C = \Big(\mathcal{T}(\frac{1}{2}$  $- \frac{h}{2}$  $), \mathcal{T}($  $)\Big)$

are mutually inverse isomorphisms. Moreover,

$\alpha_W = \Big(\mathcal{T}($  $), \mathcal{T}(\frac{1}{2}$  $+ \frac{h}{2}$  $)\Big)^t, \ \beta_W = \Big(\mathcal{T}(\frac{i}{2}$  $- \frac{ih}{2}$  $), \mathcal{T}(i$  $)\Big)$

are mutually inverse isomorphisms.

THANK YOU!

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