Twin TQFTs and Frobenius Algebras

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The category Sing-2Cob
Twin Frobenius Algebras and TQFTs
Twin TQFTs $\iff$ Twin Frobenius Algebras

Outline

- Category of singular 2-cobordisms; generators and relations
- *Twin* Frobenius Algebras
- *Twin* Topological Quantum Field Theories (twin TQFTs)
- Twin TQFTs $\iff$ Twin Frobenius Algebras
- Example
The motivation for this project has its roots mainly in:


Definition

A singular 2-cobordism is an abstract piecewise oriented smooth 2-dimensional manifold \( \Sigma \) with boundary \( \partial \Sigma = \overline{\partial^{-} \Sigma} \cup \partial^{+} \Sigma \), where \( \overline{\partial^{-} \Sigma} \) is \( \partial^{-} \Sigma \) with opposite orientation. Both \( \partial^{-} \Sigma \) and \( \partial^{+} \Sigma \) are embedded 1-dimensional closed manifolds and they are called the source and target boundaries, respectively.

Definition

The two singular 2-cobordisms \( \Sigma_1 \) and \( \Sigma_2 \) are considered equivalent, and we write \( \Sigma_1 \cong \Sigma_2 \), if there exists an orientation-preserving diffeomorphism \( \Sigma_1 \to \Sigma_2 \) which restricts to the identity on the boundary.
A singular cobordism has *singular arcs* and/or *singular circles* where orientations disagree.

Two neighboring facets of a singular cobordism are compatible oriented, inducing an orientation on the singular arc/circle that they share.

Examples:
Objects in the category Sing-2Cob

- An object in **Sing-2Cob** is diffeomorphic to a disjoint union of oriented circles and closed graphs, called bi-webs.

- An object consists of a finite sequence \( n = (n_1, n_2, \ldots, n_k) \) where \( n_j \in \{0, 1\} \).

0 = \[
\begin{array}{c}
\text{(circle)}
\end{array}
\]

1 = \[
\begin{array}{c}
\text{(bi-web)}
\end{array}
\]
A morphism $\Sigma : n \to m$ is an equivalence class of singular 2-cobordisms (induced by $\simeq$) with source boundary $n$ and target boundary $m$.

As a morphism, we “read” a cobordism from top to bottom, by convention.

Two morphisms $\Sigma_1 : n \to m$ and $\Sigma_2 : m \to k$ are composed by gluing $\Sigma_1$ on top $\Sigma_2$, along $m$:

$$\Sigma_2 \circ \Sigma_1 : n \to k$$
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Examples of morphisms

\[ \begin{align*}
\text{cozipper} & : (1) \to (0), \\
\text{zipper} & : (0) \to (1)
\end{align*} \]

\[ \circ = : (0) \to (1, 1) \]
The category **Sing-2Cob** is a symmetric monoidal category:

- The concatenation $n \amalg m := (n_1, n_2, \ldots, n_{|n|}, m_1, m_2, \ldots, m_{|m|})$ of sequences together with the *free union of singular 2-cobordisms* endows the category **Sing-2Cob** with the structure of a symmetric monoidal category.

- $|n|$ and $|m|$ above are the lengths of sequences $n$ and $m$. 
Using Morse theory, we obtain a **generators and relations description** of the category **Sing-2Cob**.

Each singular 2-cobordism can be decomposed into singular cobordisms each of which contains exactly one critical point.

The components of such a decomposition are the **generators** for the morphisms of this category.
Proposition

The monoidal category \textbf{Sing-2Cob} is generated under composition and disjoint union by the following cobordisms:
1. The oriented circle \( \bigcirc \) forms a commutative Frobenius algebra object.
2. The bi-web forms a symmetric Frobenius algebra object.

\[ \begin{align*}
& \begin{array}{c}
\text{bi-web} \\
\text{(symmetric Frobenius algebra object)}
\end{array}
\end{align*} \]
Relations

3. The “zipper” forms an algebra homomorphism.

4. The “cozipper” is dual to the zipper.
5. The image of the circle under the zipper is a central element.

6. The \textit{genus-one relation}.
Extended Frobenius Algebras

Definition

A twin Frobenius algebra $\mathbf{C} := (C, W, z, z^*)$ consists of

- a commutative Frobenius algebra $C = (C, m_C, \iota_C, \Delta_C, \epsilon_C)$,
- a symmetric Frobenius algebra $W = (W, m_W, \iota_W, \Delta_W, \epsilon_W)$,
- two morphisms $z : C \to W$ and $z^* : W \to C$

such that $z$ is a homomorphism of algebra objects and

\[
\epsilon_C \circ m_C \circ (\text{Id}_C \otimes z^*) = \epsilon_W \circ m_W \circ (z \otimes \text{Id}_W) \quad \text{(duality)}
\]

\[
m_W \circ (\text{Id}_W \otimes z) = m_W \circ \tau_W, W \circ (\text{Id}_W \otimes z) \quad \text{(centrality condition)}
\]

\[
z \circ m_C \circ \Delta_C \circ z^* = m_W \circ \tau_W, W \circ \Delta_W. \quad \text{(genus-one condition)}
\]

In particular, $z^*$ is a homomorphism of coalgebras in $C$. 
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Extended Frobenius Algebras

**Definition**

A *homomorphism of twin Frobenius algebras*

$$f : (C_1, W_1, z_1, z_1^*) \to (C_2, W_2, z_2, z_2^*)$$

consists of a pair $f = (f_1, f_2)$ of Frobenius algebra homomorphisms $f_1 : C_1 \to C_2$ and $f_2 : W_1 \to W_2$ such that $z_2 \circ f_1 = f_2 \circ z_1$ and $z_2^* \circ f_2 = f_1 \circ z_1^*$.

**Proposition**

*The category T-Frob of twin Frobenius algebras and their homomorphisms is a symmetric monoidal category.*

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Consider $\text{Mod}_R$, the category of $R$-modules and module homomorphisms, where $R$ is some commutative ring.

**Definition**

A *twin Topological Quantum Field Theory* (twin TQFT) is a symmetric monoidal functor $\text{Sing-2Cob} \rightarrow \text{Mod}_R$. A *homomorphism of twin TQFTs* is a monoidal natural transformation of such functors.

**Definition**

We denote by $\text{T-TQFT}$ the category of twin TQFTs and their homomorphisms.
The category Sing-2Cob of singular 2-cobordisms is equivalent to the symmetric monoidal category freely generated by a twin Frobenius algebra.

Corollary

The category T-Frob of twin Frobenius algebras is equivalent to the category T-TQFT of twin TQFTs (as symmetric monoidal categories).
Example

Consider the ring \( R = \mathbb{Z}[i][a, h] \), where \( i \) is so that \( i^2 = -1 \) and \( a \) and \( h \) are formal variables.

Consider \( \mathcal{A} = R[X]/(X^2 - hX - a) = \langle 1, X \rangle_R \).

Construct a twin Frobenius Algebra \( (\mathcal{A}_C, \mathcal{A}_W, z, z^*) \)

- \( \mathcal{A}_C = (\mathcal{A}, m_C, \iota_C, \Delta_C, \epsilon_C) \) is commutative Frobenius;
- \( \mathcal{A}_W = (\mathcal{A}, m_W, \iota_W, \Delta_W, \epsilon_W) \) is commutative (thus symmetric) Frobenius;
- \( z: \mathcal{A}_C \rightarrow \mathcal{A}_W, \begin{cases} z(1) = 1 \\ z(X) = X \end{cases} \) is a homomorphism of Frobenius algebra objects;
- \( z^*: \mathcal{A}_W \rightarrow \mathcal{A}_C, \begin{cases} z^*(1) = -i \\ z^*(X) = -iX \end{cases} \) is a homomorphism of Frobenius coalgebra objects, dual to \( z \).
Example

Structure maps on $\mathcal{A}_C$ and $\mathcal{A}_W$:

- **unit maps:**
  - $\iota_C: \mathbb{R} \to \mathcal{A}$, $\iota_C(1) = 1$
  - $\iota_W: \mathbb{R} \to \mathcal{A}$, $\iota_W(1) = 1$

- **counit maps:**
  - $\epsilon_C: \mathcal{A} \to \mathbb{R}$, \[
  \begin{align*}
  \epsilon_C(1) &= 0 \\
  \epsilon_C(X) &= 1
  \end{align*}
  
  - $\epsilon_W: \mathcal{A} \to \mathbb{R}$, \[
  \begin{align*}
  \epsilon_W(1) &= 0 \\
  \epsilon_W(X) &= -i
  \end{align*}
  
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Example

Structure maps on $A_C$ and $A_W$:

- **multiplication maps**: $m_{C,W}: A \otimes A \rightarrow A$

  $m_C = m_W : \begin{cases} m_C(1 \otimes X) = m_C(X \otimes 1) = X \\ m_C(1 \otimes 1) = 1, \ m_C(X \otimes X) = hX + a \end{cases}$

- **comultiplication maps**: $\Delta_{C,W}: A \rightarrow A \otimes A$

  - $\Delta_C : \begin{cases} \Delta_C(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ \Delta_C(X) = X \otimes X + a1 \otimes 1 \end{cases}$

  - $\Delta_W : \begin{cases} \Delta_W(1) = i(1 \otimes X + X \otimes 1 - h1 \otimes 1) \\ \Delta_W(X) = i(X \otimes X + a1 \otimes 1) \end{cases}$
Remark on the two structures defined on $\mathcal{A}$

- Note that $\mathcal{A}_W$ is a *twisting* of $\mathcal{A}_C$:

  i.e. comultiplication $\Delta_W$ and counit $\epsilon_W$ are obtained from $\Delta_C$ and $\epsilon_C$ by twisting them by the invertible element $-i \in \mathcal{A}$:

  \[
  \epsilon_W(x) = \epsilon_C(-ix) \quad \text{for all } x \in \mathcal{A},
  \]
  
  \[
  \Delta_W(x) = \Delta_C((-(−i)^{-1})x) = \Delta_C(ix) \quad \text{for all } x \in \mathcal{A}.
  \]

- Kadison showed that twisting by invertible elements of $\mathcal{A}$ is the only way to modify the counit and comultiplication in Frobenius extensions.
Example

Twin Frobenius algebra \((\mathcal{A}_C, \mathcal{A}_W, z, z^*)\) gives rise to a twin TQFT:

\[ \mathcal{T} : \text{Sing-2Cob} \rightarrow \text{Mod}_R \]

On objects, \(\mathcal{T}\) is defined as follows:

- \(\mathcal{T}(\text{empty manifold}) = R\).
- \(\mathcal{T}(\mathbf{n}) = \mathcal{A}_i^\otimes k\), for \(\mathbf{n} = (n_1, n_2, \ldots, n_k)\) a \(k\)-component object, where

\[
\begin{align*}
\mathcal{A}_i &= \mathcal{A}_C \quad \text{for} \quad n_i = 0 = \bigcirc, \\
\mathcal{A}_i &= \mathcal{A}_W \quad \text{for} \quad n_i = 1 = \leftrightarrow.
\end{align*}
\]
On the generating morphisms, $\mathcal{T}$ is defined as follows:

$$
\begin{aligned}
\mathcal{T}: & \rightarrow \Delta_C & \mathcal{T}: & \rightarrow m_C & \mathcal{T}: & \rightarrow \iota_C & \mathcal{T}: & \rightarrow \epsilon_C \\
\mathcal{T}: & \rightarrow \Delta_W & \mathcal{T}: & \rightarrow m_W & \mathcal{T}: & \rightarrow \iota_W & \mathcal{T}: & \rightarrow \epsilon_W \\
\mathcal{T}: & \rightarrow z & \mathcal{T}: & \rightarrow \text{Id}_{A_C} & \mathcal{T}: & \rightarrow z^* & \mathcal{T}: & \rightarrow \text{Id}_{A_W}
\end{aligned}
$$

**Remark:** $(A_C, A_W, z, z^*)$ is ‘almost’ a twin Frobenius algebra (all properties of such an algebra are satisfied except for the “genus-one condition” which holds up to a sign).
THANK YOU!