Homework # 10 Solutions

Math 111, Fall 2014
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Prove each of the following with either induction, strong induction, or proof by smallest counterexample.

1. For every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   Solution:

   Proof. We will prove this using induction.

   (1) If $n = 1$, then the statement is $1^2 = \dfrac{1(1+1)(2(1)+1)}{6} = 1$, which is true.

   (2) Let $k \geq 1$. Assume that $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$.

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1)}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6}. \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

   Therefore, it follows by induction that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every positive integer n. □

2. If $n \in \mathbb{N}$, then $\dfrac{1}{2!} + \dfrac{2}{3!} + \cdots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$.

Solution:

Proof. We will prove this using induction.

(1) If $n = 1$, then the statement is $\dfrac{1}{(1+1)!} = 1 - \dfrac{1}{(1+1)!} = \dfrac{1}{2}$, which is true.

(2) Let $k \geq 1$. Assume that $\dfrac{1}{2!} + \dfrac{2}{3!} + \cdots + \dfrac{k}{(k+1)!} = 1 - \dfrac{1}{(k+1)!}$.

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{((k+1)+1)!} &= 1 - \frac{1}{(k+1)!} + \frac{k+1}{((k+1)+1)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)(k+1)!} \\
&= 1 - \frac{k+2}{(k+2)(k+1)!} + \frac{k+1}{(k+2)(k+1)!} \\
&= 1 - \frac{1}{(k+2)!} \\
&= 1 - \frac{1}{((k+1)+1)!}
\end{aligned}$$

Therefore, it follows by induction that $\dfrac{1}{2!} + \dfrac{2}{3!} + \cdots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$ for every $n \in \mathbb{N}$. □

3. For any integer $n \geq 0$, it follows that $9 \mid (4^{3n} + 8)$.

Solution:

Proof. We prove this using induction.

(1) If $n = 0$, then the statement is $9 \mid (4^{3(0)} + 8)$, or $9 \mid 9$, which is true.

(2) Let $k \geq 0$. Assume that $9 \mid (4^{3k} + 8)$. We need to show that $9 \mid (4^{3(k+1)} + 8)$. Since $9 \mid (4^{3k} - 1)$, there is an integer x such that $4^{3k} + 8 = 9x$, so $4^{3k} = 9x - 8$.

Since $4^{3(k+1)} = 4^{3k}4^3$, we have that

$$\begin{aligned}
4^3 4^{3k} &= 4^3(9x-8) \\
4^{3k+3} &= 4^3(9x-8) \\
4^{3(k+1)} + 8 &= 4^3(9x-8) + 8 \\
&= 64(9x) - 64(8) + 8 \\
&= 9(64x) - 63(8) \\
&= 9(64x) - 9(56) \\
&= 9(64x - 56).
\end{aligned}$$

Since $64x - 56$ is an integer, $9 \mid (4^{3(k+1)} + 8)$.

Therefore, it follows by induction that $9 \mid (4^{3n}+8)$ for every non-negative integer n. $\square$

4. Suppose that $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Then

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Solution:

Proof. For this proof, we need to use strong induction.

(1) When $n = 2$, the statement is $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$. We need to prove that this is true. So,

$$\begin{aligned}
\overline{A_1 \cup A_2} &= \{x : (x \in U) \wedge (x \notin (A_1 \cup A_2))\} \quad \text{(definition of complement)} \\
&= \{x : (x \in U) \wedge \sim (x \in A_1 \cup A_2)\} \\
&= \{x : (x \in U) \wedge \sim ((x \in A_1) \vee (x \in A_2))\} \quad \text{(definition of union)} \\
&= \{x : (x \in U) \wedge (\sim (x \in A_1)) \wedge (\sim (x \in A_2))\} \quad \text{(DeMorgan's law)} \\
&= \{x : (x \in U) \wedge ((x \notin A_1) \wedge (x \notin A_2)) \\
&= \{x : ((x \in U) \wedge (x \notin A_1)) \wedge ((x \in U) \wedge (x \notin A_2))\} \quad \text{(distributive property)} \\
&= \{x : (x \in U) \wedge (x \notin A_1)\} \cap \{x : (x \in U) \wedge (x \notin A_2)\} \quad \text{(definition of intersection)} \\
&= \overline{A_1} \cap \overline{A_2} \quad \text{(definition of complement).}
\end{aligned}$$

(2) Let $k \geq 2$. Assume that the statement is true if it involves at most k sets. Then

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k \cup A_{k+1}} &= \overline{A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup (A_k \cup A_{k+1})} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cup \overline{A_k \cup A_{k+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k-1}} \cap \overline{A_k} \cap \overline{A_{k+1}}
\end{aligned}$$

$\square$

5. For every natural number n, it follows that $2^n + 1 \leq 3^n$.

   Solution:

   Proof. We will use proof by induction for this one.

   (1) For $n = 1$, we have $2^1 + 1 = 3$ and $3^1 = 3$. Therefore, the statement is true.

   (2) Let $k \geq 1$ and assume that $2^k + 1 \leq 3^k$. Then

   $$\begin{aligned} 2^{k+1} + 1 &= 2^k 2 + 1 \\ &\leq 2 \cdot 2^k + 2 \\ &= 2(2^k + 1) \\ &\leq 2(3^k) \\ &\leq 3(3^k) \\ &= 3^{k+1}. \end{aligned}$$

   Therefore, $2^{k+1} + 1 \leq 3^{k+1}$.

   So, by induction $2^n + 1 \leq 3^n$ for all natural numbers n. □

6. Prove that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$ for every $n \in \mathbb{N}$.

   Solution:

   Proof. We will prove this by induction.

   (1) For $n = 1$, we have $1^2 = 1 = 1^3$, which is clearly true. Therefore, the statement is true for $n = 1$.

   (2) Assume that the statement is true for some $k \geq 1$; i.e., assume that $(1+2+\cdots+k)^2 = 1^3 + 2^3 + \cdots + k^3$. Then, we have

   $$\begin{aligned} (1 + 2 + \cdots + k + (k+1))^2 &= ((1 + 2 + \cdots + k) + (k+1))^2 \\ &= (1 + 2 + \cdots + k)^2 + 2(k+1)(1 + 2 + \cdots + k) + (k+1)^2 \\ &= (1 + 2 + \cdots + k)^2 + 2(k+1)\frac{k(k+1)}{2} + (k+1)^2 \quad \text{(class notes)} \\ &= 1^3 + 2^3 + \cdots + k^3 + k(k+1)^2 + (k+1)^2 \\ &= 1^3 + 2^3 + \cdots + k^3 + (k+1)(k+1)^2 \\ &= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3. \end{aligned}$$

   Therefore, by induction $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$ for every $n \in \mathbb{N}$. □

7. Prove that $\displaystyle\sum_{k=s}^{N}\binom{k}{s}=\binom{N+1}{s+1}$ for all natural numbers s and N such that $N \geq s$.
Hint: Prove this by induction on N. You may find an equality from Section 3.4 useful, as well.

Solution:

Proof. We will prove this by induction on N.

(1) For $N=1$, since $s \leq N$, it must be true that $s=1$. Then

$$\sum_{k=s}^{N}\binom{k}{s}=\sum_{k=1}^{1}\binom{k}{s}=\binom{1}{1}=1=\binom{2}{2}=\binom{N+1}{s+1}.$$

(2) Assume that the statement is true for some $N=n$, where $n \geq 1$; i.e., assume that

$$\sum_{k=s}^{n}\binom{k}{s}=\binom{n+1}{s+1}.$$

Then, we have

$$\begin{aligned}\sum_{k=s}^{n+1}\binom{k}{s}&=\sum_{k=s}^{n}\binom{k}{s}+\binom{n+1}{s}\\&=\binom{n+1}{s+1}+\binom{n+1}{s}\\&=\binom{n+2}{s+1}\quad\left(\text{since }\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}\right).\end{aligned}$$

Therefore, by induction, $\displaystyle\sum_{k=s}^{N}\binom{k}{s}=\binom{N+1}{s+1}$ for all non-negative integers s and N such that $s \leq N$. $\square$

8. Let F_n be the n^{th} term of the Fibonacci sequence. Then $\displaystyle\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$.

Solution:

Proof. We will prove this by induction.

(1) If $n=1$, then we have $\displaystyle\sum_{k=1}^{n}F_n^2=F_1=1$ and $F_1\cdot F_2=1\cdot 1=1=F_1$. So, the statement is true if $n=1$.

(2) Suppose that the statement is true for some $N \geq 1$. Then $\sum_{k=1}^{N} F_k^2 = F_N F_{N+1}$. Then we have

$$\begin{aligned}\sum_{k=1}^{N+1} F_k^2 &= \sum_{k=1}^{N} F_k^2 + F_{N+1}^2 \\ &= F_N F_{N+1} + F_{N+1} F_{N+1} \\ &= F_{N+1}(F_N + F_{N+1}) \\ &= F_{N+1} F_{N+2}.\end{aligned}$$

Therefore, by induction, $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ for all integers n where $n \geq 1$. $\square$