Homework # 11 Solutions

Math 111, Fall 2014
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1. Define a relation R on $\mathbb{Z}$ by $x\,R\,y$ if $x \cdot y \geq 0$. Prove or disprove the following:

   (a) R is reflexive;

   (b) R is symmetric;

   (c) R is transitive.

   Solution:

   (a) Relation R is reflexive, because if $n \in \mathbb{Z}$, then $n \cdot n = n^2 \geq 0$, so $n\,R\,n$.

   (b) Relation R is symmetric, because if $m, n \in \mathbb{Z}$ such that $m\,R\,n$, then $n \cdot m = m \cdot n \geq 0$, and so $n\,R\,m$.

   (c) Relation R is not transitive, because $-1\,R\,0$ and $0\,R\,1$, but $-1 \not R\ 1$.

2. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

   (a) reflexive and symmetric, but not transitive;

   (b) symmetric and transitive, but not reflexive;

   (c) symmetric, but neither transitive nor reflexive.

   You must prove that your relation satisfies the stated conditions.

   Solution:

   (a) The relation $R_1 = \{(1,1), (2,2), (3,3), (4,4), (2,1), (1,2), (3,2), (2,3)\}$ is reflexive and symmetric, but not transitive.

   - Relation R_1 is reflexive, because we can see that $a\,R_1\,a$ for $a = 1, 2, 3, 4$, or for all $a \in A$.
   - Relation R_1 is symmetric, because if $a, b \in A$ such that $a\,R_1\,b$, then $a = 1, 2, 3,$ or 4 and $b = 1, 3, 4,$ or 4 and for each of these, $b\,R_1\,a$.
   - Relation R_1 is not transitive, because $1\,R_1\,2$ and $2\,R_1\,3$, but $1 \not R_1\, 3$.

   (b) The relation $R_2 = \{(1,2), (2,1), (2,2), (1,1)\}$ is symmetric and transitive, but not reflexive.

   - Relation R_2 is not reflexive because $3 \not R_2\, 3$.

- Relation R_2 is symmetric because the only $a, b \in A$ with $a \neq b$ for which $a\,R_2\,b$ is $a = 1$ or 2 and $b = 1$ or 2. Since $1\,R_2\,2$ and $2\,R_2\,1$, R_2 is symmetric.
- Relation R_2 is transitive. We have that $1\,R_2\,2$ and $2\,R_2\,1$ implies $1\,R_2\,1$; $1\,R_2\,2$ and $2\,R_2\,2$ implies $1\,R_2\,2$; $2\,R_2\,1$ and $1\,R_2\,2$ implies $2\,R_2\,2$; $2\,R_2\,1$ and $1\,R_2\,1$ implies $2\,R_2\,1$; $1\,R_2\,1$ and $1\,R_2\,2$ implies $1\,R_2\,2$; and $2\,R_2\,2$ and $2\,R_2\,1$ implies $2\,R_2\,1$. So, for all $a, b, c \in A$ such that that $a\,R_2\,b$ and $b\,R_2\,c$, we have $a\,R_2\,c$. Therefore R_3 is transitive.

(c) The relation $R_3 = \{(1,2), (2,1)\}$ is symmetric, but neither reflexive nor transitive.

- Relation R_3 is not reflexive because $1 \not R_3\, 1$.
- Relation R_3 is symmetric because the only $a, b \in A$ for which $a\,R_3\,b$ are $a = 1$ or 2 or $b = 1$ or 2. Since $1\,R_3\,2$ and $2\,R_3\,1$, we have that for every $a, b \in A$ for which $a\,R_3\,b$, $b\,R_3\,a$. Therefore R_3 is symmetric.
- Relation R_3 is not transitive because $1\,R_3\,2$ and $2\,R_3\,1$, but $1 \not R_3\, 1$.

3. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a\,R\,c$, $c\,R\,d$, $d\,R\,g$, and $b\,R\,f$. If there are three distinct equivalence classes that result from R, then determine these equivalence classes and determine all elements of R.

Solution: Since R is reflexive, we have that $a\,R\,a$, $b\,R\,b$, $c\,R\,c$, $d\,R\,d$, $e\,R\,e$, $f\,R\,f$. Also, since R is symmetric, we know that $c\,R\,a$, $d\,R\,c$, $g\,R\,d$, and $f\,R\,b$. Finally, since R is transitive, we have that $a\,R\,d$, $a\,R\,g$, and $c\,R\,g$, and by symmetry, $d\,R\,a$, $g\,R\,a$, and $g\,R\,c$. The three equivalence classes are

$$[a] = \{x \in A : x\,R\,a\} = \{a, c, d, g\} = [c] = [d] = [g];$$
$$[b] = \{x \in A : x\,R\,b\} = \{b, f\} = [f]; \text{ and}$$
$$[e] = \{x \in A : x\,R\,e\} = \{e\}.$$

We Relation R is given by

$$R = \{(a,a), (a,c), (a,d), (a,g), (c,a), (c,c), (c,d), (c,g), (d,a), (d,c), (d,d), (d,g), (g,a), (g,c), (g,d), (g,g), (b,b), (b,f), (f,b), (f,f), (e,e)\}.$$

4. Define a relation R on $\mathbb{Z}$ as $x\,R\,y$ if and only if $x^2 + y^2$ is even. Prove R is an equivalence relation and determine its distinct equivalence classes.

Solution: To show that R is an equivalnce relation, we must show that R is reflexive, symmetric, and transitive. If $x \in \mathbb{Z}$, then $x^2 + x^2 = 2x^2$ and since $x^2 \in \mathbb{Z}$, $x^2 + x^2$ is even, and so $x\,R\,x$ and R is reflexive. Next, suppose that $x, y \in \mathbb{Z}$ such that $x\,R\,y$. Then $x^2 + y^2 = y^2 + x^2$ is even and so $y\,R\,x$, and R is symmetric. Finally, suppose that $x, y, z \in \mathbb{Z}$ such that $x\,R\,y$ and $y\,R\,z$. Then $x^2 + y^2$ is even and $y^2 + z^2$ is even. So, $x^2 + y^2 = 2a$ and $y^2 + z^2 = 2b$ for some integers a and b. Then we have that $x^2 = 2a - y^2$

and $z^2 = 2b - y^2$, so $x^2 + z^2 = 2a - y^2 + 2b - y^2 = 2a + 2b - 2y^2 = 2(a + b - y^2)$. Since $a + b - y^2 \in \mathbb{Z}$, $x^2 + z^2$ is even and so $x \, R \, z$. Therefore, R is transitive.

The distinct equivalence classes of R are

$$[0] = \{x \in \mathbb{Z} : x \text{ is even}\}$$
$$[1] = \{x \in \mathbb{Z} : x \text{ is odd}\}.$$

We see that these are the only distinct equivalence classes of R because we have proven that for the sum of two numbers to be even, they both must have the same parity, and for $x \in \mathbb{Z}$, x^2 is even/odd if and only if x is even/odd. So, $[0] = \{x \in \mathbb{Z} : x \, R \, 0\} = \{x \in \mathbb{Z} : x^2 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is even}\}$. Similarly, $[2] = \{x \in \mathbb{Z} : x \, R \, 2\} = \{x \in \mathbb{Z} : x^2 + 2^2 \text{ is even}\} = \{x \in \mathbb{Z} : x^2 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is even}\} = [0]$. A similar argument will hold for $[1] = [3] = \cdots$.

5. Prove or disprove. If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

   Solution: This statement is true.

   Proof. Let R and S be equivalence relations. Then R and S are both reflexive, symmetric, and transitive. We must show that $R \cap S$ is reflexive, symmetric, and transitive. First, let $a \in A$. Then $a \, R \, a$ since R is reflexive and $a \, S \, a$ since S is reflexive. Then, by definition $(a, a) \in R$ and $(a, a) \in S$. Therefore, $(a, a) \in R \cap S$, and $a \, (R \cap S) \, a$, so $R \cap S$ is reflexive. Next, suppose that $a \, (R \cap S) \, b$ for some $a, b \in A$. By definition, we have that $(a, b) \in R \cap S$. Therefore, $(a, b) \in R$ and $(a, b) \in S$. Since R and S are symmetric, we have that $(b, a) \in R$ and $(b, a) \in S$. Therefore, $(b, a) \in R \cap S$, or $b \, (R \cap S) \, a$, and $R \cap S$ is symmetric. Finally, let $a \, (R \cap S) \, b$ and $b \, (R \cap S) \, c$. Then, $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$, and so $(a, b) \in R$ and $(a, b) \in S$, and $(b, c) \in R$ and $(b, c) \in S$. Since R and S are transitive, we have $(a, c) \in R$ and $(a, c) \in S$, so $(a, c) \in R \cap S$, or $a \, (R \cap S) \, c$, and $R \cap S$ is transitive. Since $R \cap S$ is reflexive, symmetric, and transitive, $R \cap S$ is an equivalence relation. □

6. Describe the partition of $\mathbb{Z}$ resulting from the equivalence relation $\equiv \pmod 3$.

   Solution: First,

$$[0] = \{x \in \mathbb{Z} : x \equiv 0 \pmod 3\} = \{x \in \mathbb{Z} : 3 \mid x\} = \{x \in \mathbb{Z} : x = 3k, k \in \mathbb{Z}\}.$$

   In other words, $[0]$ is the set of all positive and negative multples of 3, so $[0] = [3] = [-3] = [6] = \ldots$. Similarly,

$$[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 3\} = \{x \in \mathbb{Z} : 3 \mid (x - 1)\} = \{x \in \mathbb{Z} : x - 1 = 3k, k \in \mathbb{Z}\}.$$

So, $[1]$ is the set of all $x \in \mathbb{Z}$ such that $x-1$ is a multiple of 3, and we have $[1] = [-1] = [4] = [-4] = \ldots$. Finally,

$$[2] = \{x \in \mathbb{Z} : x \equiv 2 \pmod 3\} = \{x \in \mathbb{Z} : 3 \mid (x-2)\} = \{x \in \mathbb{Z} : x - 2 = 3k, k \in \mathbb{Z}\}.$$

So, $[2]$ is the set of all $x \in \mathbb{Z}$ such that $x-2$ is a multiple of 3, and we have $[2] = [-2] = [5] = [-5] = \ldots$. Therefore, the partition of $\mathbb{Z}$ resulting from $\equiv \pmod 3$ is

$$\{[0],[1],[2]\} = \{\{\ldots,-6,-3,0,3,\ldots\},\{\ldots,-7,-4,-1,1,4,7,\ldots\},\{\ldots,-8,-5,-2,2,5,8,\ldots\}\}.$$

7. Write the addition and multiplication tables for $\mathbb{Z}_8$.

   Solution:

Addition and multiplication tables for Z_8.

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[7]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[7]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[7]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[7]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[7]	[0]	[1]	[2]	[3]	[4]	[5]
[7]	[7]	[0]	[1]	[2]	[3]	[4]	[5]	[6]

·	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[2]	[0]	[2]	[4]	[6]	[0]	[2]	[4]	[6]
[3]	[0]	[3]	[6]	[1]	[4]	[7]	[2]	[5]
[4]	[0]	[4]	[0]	[4]	[0]	[4]	[0]	[4]
[5]	[0]	[5]	[2]	[7]	[4]	[1]	[6]	[3]
[6]	[0]	[6]	[4]	[2]	[0]	[6]	[4]	[2]
[7]	[0]	[7]	[6]	[5]	[4]	[3]	[2]	[1]

8. Prove or disprove. If $[a],[b] \in \mathbb{Z}_6$ and $[a][b] = [0]$, then either $[a] = [0]$ or $[b] = [0]$.

   Solution: This statement is false. Let $[a] = [2]$ and $[b] = [3]$. Then $[2][3] = [0]$, since $[2][3] = [2 \cdot 3] = [6] = [0]$ but $[2] \neq [0]$ and $[3] \neq [0]$.