Homework # 12 Solutions

Math 111, Fall 2014
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1. Give an example of a relation from $\mathbb{Z}$ to $\{0, 1\}$ that is **not** a function.

   Solution: Let $R = \{\dots, (-2,-2), (-1,-1), (0,0), (0,1), (1,1), (2,2), \dots\}$. Relation R is not a function because if $x = 0$, the pairs $(x, 1)$ and $(x, 2)$ are in R.

2. Suppose $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ is defined as $f = \{(x, 4\sqrt{x} - 5) : x \in \mathbb{Z}\}$. State the domain, codomain, and range of f.

   Solution: The domain of f is $\mathbb{N} \cup \{0\}$ and the codomain of f is $\mathbb{Z}$. The smallest value of $x \in \mathbb{N} \cup \{0\}$ is $x = 0$, and $f(0) = -5$. Therefore, the smallest value of $f(x)$ is -5. As x gets larger, $4\sqrt{x} - 5$ increases, so the range of f would be $[-5, \infty)$ if the codomain of f were $\mathbb{R}$. However, the codomain of f is $\mathbb{Z}$. Since $4\sqrt{x} - 5 \in \mathbb{Z}$ if and only if $\sqrt{x} \in \mathbb{Z}$, we have that $x \in \{0, 1, 4, 9, 16, \dots\}$. Therefore, the range of f is

   $$\{-5, -1, 3, 7, 11, \dots\} = \{-5 + 4n : n \in \mathbb{N} \cup \{0\}\}.$$

3. Consider functions from $\mathbb{Z}$ to $\mathbb{Z}$. Give an example of

   (a) a function that is injective but not surjective;

   (b) a function that is surjective but not injective; and

   (c) a function that is neither injective nor surjective.

   For each example, prove that your function satisfies the given property.

   Solution:

   (a) The function $f = \{(x, 3x) : x \in \mathbb{Z}\}$ is injective but not surjective.

   Proof. Let $a, b \in \mathbb{Z}$. Then if $f(a) = f(b)$, we have $3a = 3b$. Dividing both sides by 3 gives $a = b$. Therefore, f is injective.

   Consider $2 \in \mathbb{Z}$. Then $3x = 2$ only if $x = \dfrac{2}{3} \notin \mathbb{Z}$. Therefore, there is a point $y \in \mathbb{Z}$ for which there is no x such that $f(x) = y$ and so f is not surjective. $\square$

(b) The function

$$g = \begin{cases} x & \text{if } x \geq 1, \\ x+1 & \text{if } x \leq 0. \end{cases}$$

is surjective but not injective.

Proof. We have that $g(0) = 1$ and $g(1) = 1 = g(0)$, but $0 \neq 1$. Therefore, g is not injective

Let $y \in \mathbb{Z}$. If $y = 1$, then we know that $g(0) = g(1) = 1$. If $y > 1$, then choose $x = y > 1$. Since $x = y$, $g(x) = g(y) = y$. If $y \leq 0$, then let $x = y - 1 \leq 0$. So $g(x) = g(y-1) = (y-1)+1 = y$. Therefore, for any $y \in \mathbb{Z}$, there exists x such that $g(x) = y$, and so g is surjective. □

(c) The function $h = \{(x, x^2 - 4x + 4) : x \in \mathbb{Z}\}$ is neither injective nor surjective.

Proof. We have that $h(1) = 1^2 - 4(1) + 4 = 1$ and $h(3) = 3^2 - 4(3) + 4 = 1 = h(1)$, but $1 \neq 3$. Therefore, h is not injective.

Next, let $y = -1$. There is no $x \in \mathbb{Z}$ such that $x^2 - 4x + 4 = -1$ (since $x^2 - 4x + 4 = (x-2)^2$). Therefore, h is not surjective. □

4. Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.

Solution: *Side work:* To show that f is surjective, we need to show that for any $y \in \mathbb{R} - \{1\}$, we can find an x such that

$$\left(\frac{x+1}{x-1}\right)^3 = y.$$

Take the cube root of both sides and multiply by $x - 1$. This gives

$$x + 1 = y^{\frac{1}{3}}(x-1).$$

Now, we solve for x.

$$\begin{aligned} x + 1 &= y^{\frac{1}{3}}x - y^{\frac{1}{3}} \\ x - y^{\frac{1}{3}}x &= -y^{\frac{1}{3}} - 1 \\ x\left(1 - y^{\frac{1}{3}}\right) &= -\left(y^{\frac{1}{3}} + 1\right) \\ x &= -\frac{\left(y^{\frac{1}{3}} + 1\right)}{1 - \left(y^{\frac{1}{3}}\right)} \\ &= \frac{\left(y^{\frac{1}{3}} + 1\right)}{\left(y^{\frac{1}{3}} - 1\right)}. \end{aligned}$$

Proof. First, we will show that f is injective. Suppose that there exist $a, b \in \mathbb{R} - \{1\}$ such that $f(a) = f(b)$. Then

$$\left(\frac{a+1}{a-1}\right)^3 = \left(\frac{b+1}{b-1}\right)^3.$$

Take the cube root of both sides to obtain

$$\frac{a+1}{a-1} = \frac{b+1}{b-1}.$$

Multiply both sides by $(a-1)(b-1)$:

$$\begin{aligned}(a+1)(b-1) &= (b+1)(a-1)\\ ab+b-a-1 &= ab+a-b-1.\end{aligned}$$

Subtract ab from and add 1 to both sides:

$$b - a = a - b.$$

Finally, add $a+b$ to both sides to obtain

$$2b = 2a.$$

Dividing by 2 gives $a = b$. Therefore, f is injective.

Next, we will show that f is surjective. Let $y \in \mathbb{R} - \{1\}$. Then, set $x = \dfrac{\left(y^{\frac{1}{3}}+1\right)}{\left(y^{\frac{1}{3}}-1\right)}$.

We thus have

$$\begin{aligned}f(x) &= \left(\frac{\frac{\left(y^{\frac{1}{3}}+1\right)}{\left(y^{\frac{1}{3}}-1\right)}+1}{\frac{\left(y^{\frac{1}{3}}+1\right)}{\left(y^{\frac{1}{3}}-1\right)}-1}\right)^3\\ &= \left(\frac{\frac{\left(y^{\frac{1}{3}}+1+\left(y^{\frac{1}{3}}-1\right)\right)}{\left(y^{\frac{1}{3}}-1\right)}}{\frac{\left(y^{\frac{1}{3}}+1-\left(y^{\frac{1}{3}}-1\right)\right)}{\left(y^{\frac{1}{3}}-1\right)}}\right)^3\\ &= \left(\frac{2y^{\frac{1}{3}}}{y^{\frac{1}{3}}-1}\cdot\frac{y^{\frac{1}{3}}-1}{2}\right)^3\\ &= \left(y^{\frac{1}{3}}\right)^3\\ &= y.\end{aligned}$$

Since $f(x) = y$ and y was arbitrary, f is surjective. $\square$

5. Suppose that A, B, and C are nonempty sets and $f : A \to B$ and $g : B \to C$.

   (a) Prove or disprove. If $g \circ f$ is injective, then f is injective.

   (b) Prove or disprove. If $g \circ f$ is injective, then g is injective.

   Solution:

   (a) This statement is true.

   Proof. Suppose $g \circ f$ is injective. Let $f(x) = f(y)$ for some $x, y \in A$. Then $g(f(x)) = g(f(y))$, and so it follows that $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is injective, $x = y$. Therefore, f is injective. $\square$

   (b) This statement is false.

   Counterexample. Let $A = \{1, 2, 3\}$, $B = \{-1, -2, -3, -4\}$, and $C = \{0, 5, 10\}$. Define $f : A \to B$ by

$$f = \{(1, -1), (2, -2), (3, -3)\}$$

   and $g : B \to C$ by

$$g = \{(-1, 0), (-2, 5), (-3, 10), (-4, 10)\}.$$

   Then $g \circ f : A \to C$ is given by

$$g \circ f = \{(1, 0), (2, 5), (3, 10)\}.$$

   We see that $g \circ f : A \to C$ is injective, but $g : B \to C$ is not. $\square$

6. Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m, n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m, m)$. Find formulas for $g \circ f$ and $f \circ g$.

   Solution:

$$\begin{aligned}
(g \circ f)(m, n) &= g(f(m, n)) \\
&= g(m + n) \\
&= (m + n, m + n). \\
(f \circ g)(m) &= f(g(m)) \\
&= f(m, m) \\
&= m + m = 2m.
\end{aligned}$$