Homework # 4 Solutions

Math 111, Fall 2014
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1. Write a truth table for the following statements.

   (a) $(Q \vee R) \iff (R \wedge Q)$

   (b) $\sim (P \wedge Q) \wedge (\sim P)$

   (c) $P \vee (Q \wedge \sim R)$

   (d) $(P \implies Q) \implies (\sim P)$

   Solution:

   (a) $(Q \vee R) \iff (R \wedge Q)$

Q	R	$Q \vee R$	$R \wedge Q$	$(Q \vee R) \iff (R \wedge Q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

   (b) $\sim (P \wedge Q) \wedge (\sim P)$

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$	$\sim P$	$\sim (P \wedge Q) \wedge (\sim P)$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	T	T	T

   (c) $P \vee (Q \wedge \sim R)$

P	Q	R	$\sim R$	$Q \wedge \sim R$	$P \vee (Q \wedge \sim R)$
T	T	T	F	F	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

(d) $(P \implies Q) \implies (\sim P)$

P	Q	$P \implies Q$	$\sim P$	$(P \implies Q) \implies (\sim P)$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

2. Suppose the statement $((P \wedge Q) \vee R) \implies (R \vee S)$ is false. Without using a truth table, determine the truth values of P, Q, R, and S.

   Solution: If the implication is false, that means that the statement on the left is true and the statement on the right is false. The statement $(R \vee S)$ is only false if both R and S are false. Since $(P \wedge Q) \vee R$ is true, this means that $P \wedge Q$ must be true, which is only the case if both P and Q are true. Therefore, we have that P is true, Q is true, R is false, and S is false.

3. Use truth tables to show that the following statements are equivalent.

   (a) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

   (b) $\sim P \iff Q = (P \implies \sim Q) \wedge (\sim Q \implies P)$

   Solution:

   (a) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

(b) $\sim P \iff Q = (P \implies \sim Q) \wedge (\sim Q \implies P)$

P	Q	$\sim P$	$\sim P \iff Q$	$\sim Q$	$P \implies (\sim Q)$	$\sim Q \implies P$	$(P \implies \sim Q) \wedge (\sim Q \implies P)$
T	T	F	F	F	F	T	F
T	F	F	T	T	T	T	T
F	T	T	T	F	T	T	T
F	F	T	F	T	T	F	F

4. Determine whether or not the following pairs of statements are logically equivalent.

   (a) $\sim (P \implies Q)$ and $P \wedge \sim Q$

   (b) $(\sim Q) \implies (P \wedge \sim P)$ and Q

   (c) $(P \wedge Q) \iff P$ and $P \implies Q$

   (d) $\sim (P \vee Q)$ and $(\sim P) \vee (\sim Q)$

   Solution:

   (a) $\sim (P \implies Q)$ and $P \wedge \sim Q$

P	Q	$P \implies Q$	$\sim (P \implies Q)$	$\sim Q$	$P \wedge (\sim Q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Since the truth tables of the two statements are identical, they are logically equivalent.

(b) $(\sim Q) \implies (P \wedge \sim P)$ and Q

P	Q	$\sim Q$	$P \wedge \sim P$	$(\sim Q) \implies (P \wedge \sim P)$	Q
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F

Since the truth tables of the two statements are identical, they are logically equivalent.

(c) $(P \wedge Q) \iff P$ and $P \implies Q$

P	Q	$P \wedge Q$	$(P \wedge Q) \iff P$	$P \implies Q$
T	T	T	T	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Since the truth tables of the two statements are identical, they are logically equivalent.

(d) $\sim (P \vee Q)$ and $(\sim P) \vee (\sim Q)$

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	F	T
F	F	F	T	T	T	T

Since the truth tables of the two statements are not identical, they are not logically equivalent.

5. Write the following as English sentences. State whether they are true or false.

(a) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x^n \geq 0$

(b) $\exists n \in \mathbb{N}, \forall X \in \mathcal{P}(\mathbb{N}), |X| < n$

(c) $\forall n \in \mathbb{Z}, \exists X \subseteq \mathbb{N}, |X| = n$

(d) $\forall X \in \mathcal{P}(\mathbb{N}), X \subseteq \mathbb{R}$

Solution:

(a) For all real numbers x, there exists a natural number n such that $x^n \geq 0$. True. If $n = 2$ (or any even number), $x^n \geq 0$.

(b) There exists a natural number n such that for all sets X in the power set of the natural numbers, $|X| < n$. False. The set $\mathbb{N} \in \mathcal{P}(\mathbb{N})$, and cardinality is not defined for sets with infinitely many elements.

(c) For all integers n, there exists a subset X of the natural numbers such that $|X| = n$. False. Since cardinality is defined to be a nonnegative number, if $n < 0$, this is not true.

(d) For all sets X in the power set of $\mathbb{N}$, X is a subset of $\mathbb{R}$. True, because $\mathbb{N} \subseteq \mathbb{R}$, so if $X \subseteq \mathbb{N}$, then $X \subseteq \mathbb{R}$.

6. Translate each of the following sentences into symbolic logic.

(a) The number x is positive and the number y is positive.

(b) For every positive number ϵ there is a positive number M for which $|f(x) - b| < \epsilon$, whenever $x > M$.

(c) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

(d) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

(e) If $\sin x < 0$, then it is not the case that $0 \leq x \leq \pi$.

Solution:

(a) $(x > 0) \wedge (y > 0)$

(b) $\forall \epsilon > 0, \exists M > 0, (x > M) \implies |f(x) - b| < \epsilon$

(c) $\exists a, b \in \mathbb{Z}, (ab < 0) \wedge (a + b > 0)$

(d) $\forall x, y \in \mathbb{R}, (x \neq y) \implies (x^2 + y^2 > 0)$

(e) $\sin x < 0 \implies \sim (0 \leq x \leq \pi)$

7. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is T. Which of the following implies that $P(x) \implies Q(x)$ is true for all $x \in T$?

(a) $P(x) \wedge Q(x)$ is false for all $x \in T$.

(b) $Q(x)$ is true for all $x \in T$.

(c) $P(x)$ is false for all $x \in T$.

(d) $P(x) \wedge (\sim Q(x))$ is true for some $x \in T$.

(e) $(\sim P(x)) \wedge (\sim Q(x))$ is false for all $x \in T$.

Solution: The correct answer is (b) and (c).

(a) If $P(x) \wedge Q(x)$ is false for all $x \in T$, then at least one of $P(x)$ and $Q(x)$ is false. But, if $P(x)$ is true and $Q(x)$ is false, then $P(x) \wedge Q(x)$ is false and $P(x) \implies Q(x)$ is false.

(b) If $Q(x)$ is true for all $x \in T$, then $P(x) \implies Q(x)$ is true for all $x \in T$.

(c) If $P(x)$ is false for all $x \in T$, then for any truth value of $Q(x)$, we have $P(x) \implies Q(x)$ is true.

(d) If $P(x) \wedge (\sim Q(x))$ is true for some $x \in T$, then we only know that both $P(x)$ and $\sim Q(x)$ are true for some $x \in T$, which means $P(x)$ is true and $Q(x)$ is false. Therefore, for some $x \in T$, $P(x) \implies Q(x)$ is false.

(e) If $(\sim P(x)) \wedge (\sim Q(x))$ is false for all $x \in T$, then at least one of $\sim P(x)$ and $\sim Q(x)$ is false. This means that at least one of $P(x)$ and $Q(x)$ is true. If $Q(x)$ is true while $P(x)$ is false, then $(\sim P(x)) \wedge (\sim Q(x))$ is false, but $P(x) \implies Q(x)$ is also false.