# Homework # 5 Solutions

Math 111, Fall 2014 Instructor: Dr. Doreen De Leon

#### 1. Write the negation of the following sentences.

- (a) Either x = 0 or y = 0.
- (b) The integers a and b are both nonnegative.
- (c) If f is a polynomial and its degree is greater than 1, than f' is not constant.
- (d) There exists a rational number r such that  $r^2 = 2$ .
- (e) If xy is even, then either x or y is even.

(f) For every rational number r, the number  $\frac{1}{r}$  is rational.

#### Solution:

- (a) This statement means  $(x = 0) \lor (y = 0)$ . Therefore, the negation of this is  $\sim ((x = 0) \lor (y = 0))$ , which is equivalent to  $(\sim (x = 0)) \land (\sim (y = 0))$ , or: Both x and y are nonzero.
- (b) This statement can be written as (the integer a is nonnegative) ∧ (the integer b is nonnegative). Therefore, the negation of this is ~ ((the integer a is nonnegative))∧ (the integer b is nonnegative)), which is equivalent to (~ (the integer a is nonnegative))∨ (~ (the integer b is nonnegative)), or

Either the integer a is negative or the integer b is negative.

(c) This statement is  $\forall f, ((f \text{ a polynomial }) \land (f \text{ has degree greater than 1})) \implies (f' \text{ is not constant})$ . Therefore, the negation of this is  $\exists f, ((f \text{ a polynomial }) \land (f \text{ has degree greater than 1})) \land (\sim (f' \text{ is not constant})), \text{ or:}$ 

There is a polynomial f whose degree is greater than 1, and f' is constant.

(d) In symbolic form, we have  $\exists r \in \mathbb{Q}, r^2 = 2$ . We need to determine  $\sim (\exists r \in \mathbb{Q}, r^2 = 2)$ . This is equivalent to  $\forall r \in \mathbb{Q}, \sim (r^2 = 2)$ , or:

For all rational numbers  $r, r^2 \neq 2$ . (Or, there is no rational number r such that  $r^2 = 2$ .)

(e) We have  $(xy \text{ even}) \implies ((x \text{ even}) \lor (y \text{ even}))$ , so the negation of this is as follows:

$$\sim (\forall x, y, (xy \text{ even }) \implies ((x \text{ even}) \lor (y \text{ even})))$$
  
=  $\exists x, y, (xy \text{ even }) \land \sim ((x \text{ even}) \lor (y \text{ even}))$   
=  $\exists x, y, (xy \text{ even }) \land ((x \text{ not even}) \land (y \text{ not even})),$ 

or: There exist numbers x and y such that xy is even and x and y are both not even.

(f) Symbolically, this is  $\forall r \in \mathbb{Q}, \frac{1}{r} \in \mathbb{Q}$ . The negation of this,  $(\forall r \in \mathbb{Q}, \frac{1}{r} \in \mathbb{Q})$ , is  $\exists r \in \mathbb{Q}, \frac{1}{r} \notin \mathbb{Q}$ , or:

There exists a rational number r such that  $\frac{1}{r}$  is not rational.

2. Let  $n \in \mathbb{Z}$ . For which implication is its negation the following? The integer 3n + 4 is odd and 5n - 6 is even.

**Solution:** The statement is  $(3n + 4 \text{ is odd}) \land (5n - 6 \text{ is even})$ . Recall that

 $\sim (P \implies Q) = P \land (\sim Q),$ 

so the correct implication will be given by  $(3n + 4 \text{ is odd}) \implies \sim (5n - 6 \text{ is even})$ , or If 3n + 4 is odd, then 5n - 6 is odd.

3. Find the number of possible user passwords with 7 characters that consist of digits or letters of the alphabet, without repetition.

**Solution:** There are 26 letters in the alphabet and 10 one-digit numbers (including 0). Therefore, there are 36 choices for the first character of the password. Since no repetition is allowed, there are 35 choices for the second character, 34 for the third, and so on. Therefore, the total number of user passwords is given by

$$36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \cdot 30 = 42072307200.$$

So there are 42,072,307,200 (over 42 million) passwords!

4. How many 7-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7 if there is no repetition and the odd digits must appear in an unbroken sequence. (So, 1357246 and 2753146 satisfy this condition, but 7654231 does not.)

**Solution:** This means that the first four digits must be odd, or the second four digits must be odd, or the third four or the fourth four. If the first four digits are odd, then there are 4! = 24 ways to do that and the remaining three digits have 3! = 6 possible

entries. Therefore, there are (4!)(3!) = 144 ways to make 7-digit numbers where the first four digits are odd. What about if the second four digits are odd? Thenn, we have 4 options for the first digit, 3! ways of arranging the next three digits and 3! ways of arranging the final 3 digits, which is 4(3!)(3!) = (4!)(3!) = 144. A similar argument holds if the third three or fourth four digits are odd. Therefore, the total number of 7-digit numbers with the odd digits appearing in an unbroken sequence is given 4(3!)(4!) = 576.

- 5. Suppose A is a set such that |A| = 100.
  - (a) How many subsets of A have 5 elements?
  - (b) How many subsets have 10 elements?
  - (c) How many have 99 elements?

### Solution:

(a) Since |A| = 100, we simply need to find

$$\binom{100}{5} = \frac{100!}{(95!)(5!)} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5!} = 75,287,520.$$

(b) Since |A| = 100, we need to find

$$\binom{100}{5} = \frac{100!}{(90!)(10!)} = 17,310,309,456,440.$$

(c) This time, we seek

$$\binom{100}{99} = \frac{100!}{(99!)(1!)} = \frac{100}{1} = 100.$$

6. Determine

$$|\{X \in P(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}) : |X| = 4\}|.$$

Solution: We need to find

$$\binom{10}{4} = \frac{10!}{(6!)(4!)} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210.$$

So,  $|\{X \in P(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}) : |X| = 4\}| = 210.$ 

- 7. This problem concerns lists made from the symbols A, B, C, D, E, F, G, H, I.
  - (a) How many length-5 lists can be made if repetition is not allowed and the list must begin with a vowel?
  - (b) How many length-5 lists can be made if repetition is not allowed and the list must contain exactly one A?
  - (c) How many length-5 lists can be made if repetition is not allowed and the list is in alphabetical order? (For example, *BDEFI* or *ABCGH* is allowed, but *BACGH* is not allowed.)
  - (d) How many length-5 lists can be made if repetition is not allowed and the list is **not** in alphabetical order?

## Solution:

- (a) If the list must begin with a vowel, there are 3 possible choices (A, E, or I). Since no repetition is allowed, that leaves 8 choices for the second element, 7 choices for the third element, 6 choices for the fourth element, and 5 choices for the last element in the list. Therefore, there are  $3 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 5,040$  length-5 lists starting with a vowel.
- (b) If the list must contain an A, suppose that A is the first item in the list. Then, since no repetition is allowed, that leaves 8 choices for the second element, 7 choices for the third element, 6 choices for the fourth element, and 5 choices for the last element in the list. Of course, A could also be the second, third, fourth, or fifth element in the list. Therefore, there are

$$5(1 \cdot 8 \cdot 7 \cdot 6 \cdot 5) = 8,040$$

length-5 lists containing exactly one A.

(c) The answer is actually  $\begin{pmatrix} 9\\5 \end{pmatrix} = 126$ . Since there is only one ordering of the list that will be in alphabetical order (once letters are chosen), the number of lists in alphabetical order is equivalent to the number of subsets containing 5 elements that can be formed from a set of 9 elements.

An alternate way to determine this is to consider that the list must begin with A, B, C, D, or E in order to be in alphabetical order, and determine the number of possible lists for each first letter. For example, if the first letter is E, then there is only one 5-letter list in alphabetical order. If the first letter is D, there are 5 lists satisfying this condition (the second letter could be E or F, and so on). Summing the results gives 126.

(d) The total number of length-5 lists that can be made is given by

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120.$$

So, the number of length-5 lists that can be made if repetition is not allowed and the list is not in alphabetical order is

(total number of length-5 lists)-(number of length-5 lists in alphabetical order),

or 15120 - 126 = 14,994 length-5 lists satisfying the requirements.

8. Suppose a set B has the property that  $|\{X : X \in P(B), |X| = 6\}| = 28$ . What is |B|? Solution: Since

$$28 = |\{X : X \in P(B), |X| = 6\}| = {|B| \choose 6},$$

and

$$\binom{|B|}{6} = \frac{|B|!}{(|B|-6)!6!} = 28,$$

we have (after trial and error) that |B| = 8 works (since  $28 \cdot 2 = 56 = 8 \cdot 7 = \binom{8}{6}$ ).