Homework # 6 Solutions

Math 111, Fall 2014
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1. Use the binomial theorem to show that

$$\sum_{k=0}^{n} \binom{n}{k} 6^k = 7^n.$$

Solution: We can view 7^n as $(6+1)^n$. Applying the binomial theorem gives

$$\begin{aligned} 7^n &= (6+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} 6^k 1^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} 6^k, \end{aligned}$$

since $1^{n-k} = 1$.

2. This problem involves lists made from the letters T, H, E, O, R, Y, with repetition allowed.

(a) How many 4-letter lists are there that do not begin with a T or do not end in a Y?

(b) How many 4-letter lists are there in which the sequence of letters T, H, E appear consecutively?

Solution:

(a) Let $A =$ the set of 4-letter lists that do not begin with a T, and let $B =$ the set of 4-letter lists that do not end in a Y. Since the number of lists (with repetition) that do not begin with a T is $(5)(6)(6)(6) = 5 \cdot 6^3 = 1080$, $|A| = 1080$. The number of lists (with repetition) that do not end in a Y is $(6)(6)(6)(5) = 6^3 5 = 1080$, so $|B| = 1080$. We seek $|A \cup B|$, which is given by

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Therefore, we need to determine $|A \cap B|$. In other words, we need to determine the number of 4-letter lists that do not begin with a T and end with a Y, but this is given by $(5)(6)(6)(5) = 5^2 6^2 = 900$. So, there are $1080 + 1080 - 900 = 1,260$ 4-letter lists that do not begin with a T or do not end in a Y.

(b) T, H, E can be either the first three letters or the second three letters. There are $1(1)(1)6 = 6$ lists where T, H, E are the first three letters and, similarly, 6 lists where T, H, E are the second three letters. Therefore, since these two sets are mutually exclusive, there are $6 + 6 = 12$ 4-letter lists in which the sequence of letters T, H, E appear consecutively.

3. Prove that if x is an odd integer, then x^3 is odd.

Solution:

Proof. Let x be an odd integer. Then $x = 2a + 1$ for some integer a. Then

$$\begin{aligned} x^3 &= (2a+1)^3 \\ &= (2a)^3 + 3(2a)^2(1) + 3(2a)(1)^2 + (1)^3 \text{ (by the binomial theorem)} \\ &= 8a^3 + 12a^2 + 6a + 1 \\ &= 2(4a^3 + 6a^2 + 3a) + 1. \end{aligned}$$

Since $4a^3 + 6a^2 + 3a$ is an integer, x^3 is odd. $\square$

4. Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $a|c$, then $a|(b + c)$.

Solution:

Proof. Let a, b, c be integers such that $a|b$ and $a|c$. Then, $b = ma$ for some integer m and $c = na$ for some integer n. Therefore,

$$b + c = ma + na = (m + n)a.$$

Since $m + n$ is an integer, $a|(b + c)$. $\square$

5. Prove that if $x \in R$ and $0 < x < 4$, then $\dfrac{4}{x(4-x)} \geq 1$.

Solution: Side work: We want to show that $\dfrac{4}{x(4-x)} \geq 1$. This is the same as showing that $4 \geq x(4 - x) = 4x - x^2$. [You should answer why this is true.]

$$x^2 - 4x + 4 \geq 0.$$

Since $x^2 - 4x + 4 = (x - 2)^2$, things work out.

Proof. Let x be a real number such that $0 < x < 4$. Then

$$\begin{aligned}
(x-2)^2 &\geq 0 \\
x^2 - 4x + 4 &\geq 0 \\
4 &\geq 4x - x^2 \\
4 &\geq x(4-x) \\
\frac{4}{x(4-x)} &\geq 1.
\end{aligned}$$

□

6. Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

   Solution:

   Proof. There are two possibilities: n is even or n is odd.

   Case 1: n is even.

   In this case, $n = 2a$ for some integer a. Then

$$\begin{aligned}
n^2 - 3n + 9 &= (2a)^2 - 3(2a) + 9 \\
&= 4a^2 - 6a + 9 \\
&= 2(2a^2 - 3a + 4) + 1.
\end{aligned}$$

   Since $2a^2 - 3a + 4$ is an integer, $n^2 - 3n + 9$ is odd.

   Case 2: n is odd.

   In this case, $n = 2b + 1$ for some integer b. Then

$$\begin{aligned}
n^2 - 3n + 9 &= (2b+1)^2 - 3(2b+1) + 9 \\
&= 4b^2 + 4b + 1 - 6b - 3 + 9 \\
&= 4b^2 - 2b + 7 \\
&= 2(2b^2 - b + 3) + 1.
\end{aligned}$$

   Since $2b^2 - b + 3$ is an integer, $n^2 - 3n + 9$ is odd.

   These cases show that $n^2 - 3n + 9$ is odd for any integer n. □

7. Prove for every nonnegative integer n that $2^n + 6^n$ is an even integer.

   Solution:

   Proof. Suppose that n is a nonnegative integer. There are two cases to consider: $n = 0$ and $n > 0$.

Case 1: $n = 0$.

If $n = 0$, then $2^n + 6^n = 2^0 + 6^0 = 1 + 1 = 2$, which is even.

Case 2: $n > 0$.

If $n > 0$, then we have the following.

$$\begin{aligned} 2^n + 6^n &= 2^n + (2 \cdot 3)^n \\ &= 2^n + 2^n 3^n \\ &= 2 \cdot 2^{n-1} + 2 \cdot 2^{n-1} 3^n \\ &= 2(2^{n-1} + 2^{n-1} 3^n). \end{aligned}$$

Since $2^{n-1} + 2^{n-1}3^n$ is an integer, $2^n + 6^n$ is even.

These cases show that if n is a nonnegative integer, $2^n + 6^n$ is an even integer. □

8. Prove that for every two distinct integers a and b, either $\dfrac{a+b}{2} > a$ or $\dfrac{a+b}{2} > b$.

Solution:

Proof. Let a and b be distinct integers. There are two possibilities: $a > b$ or $b > a$.

Case 1: $a > b$.

Since $a > b$, we have

$$\begin{aligned} \frac{a+b}{2} &= \frac{a}{2} + \frac{b}{2} \\ &> \frac{a}{2} + \frac{a}{2} \text{ (since } a > b) \\ \implies \frac{a+b}{2} &> a. \end{aligned}$$

Case 2: $b > a$.

Since $b > a$, we have

$$\begin{aligned} \frac{a+b}{2} &= \frac{a}{2} + \frac{b}{2} \\ &> \frac{b}{2} + \frac{b}{2} \text{ (since } b > a) \\ \implies \frac{a+b}{2} &> b. \end{aligned}$$

These cases show that if a and b are distinct integers, then either $\dfrac{a+b}{2} > a$ or $\dfrac{a+b}{2} > b$. □

9. Evaluate the proof of the following proposition.

Proposition. If x and y are integers, then xy^2 has the same parity as x.

Proof. Assume, without loss of generality, that x is even. Then $x = 2a$ for some integer a. Thus,

$$xy^2 = (2a)y^2 = 2(ay^2).$$

Since ay^2 is an integer, xy^2 is even. $\square$

Solution: The problem with this proof is the assumption "without loss of generality." This proof really only shows that if x is an even integer and y is an arbitrary integer, then xy^2 is even. To show the given proposition, we need to also analyze the case where x is odd.