Homework # 7 Solutions

Math 111, Fall 2014
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1. Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

   Solution:

   Proof. (contrapositive)
   Suppose that $x \in \mathbb{R}$ such that $x \leq -1$. Then

$$\begin{aligned} x^3 - x &= x(x^2 - 1) \\ &= x(x+1)(x-1). \end{aligned}$$

   Since $x \leq -1$, we have that $x \leq 0$,

$$x + 1 \leq -1 + 1, \text{ or } x + 1 \leq 0,$$

   and $x - 1 \leq -1 - 1 = -2$, so $x - 1 \leq 0$. Since the product of three negative numbers is negative, we have that $x^3 - x \leq 0$. □

2. The product of an irrational number and a nonzero rational number is irrational.

   Solution:

   Proof. (contradiction) Suppose to the contrary, that there exist an irrational number a and a nonzero rational number b whose product is rational. Since ab is rational, we may write

$$ab = \frac{m}{n},$$

   where $m, n \in Z$ and $n \neq 0$. And, since a is a nonzero rational number, we can write

$$a = \frac{k}{l},$$

   where $k, l \in \mathbb{Z}$, $k \neq 0$, and $l \neq 0$. Then

$$\frac{k}{l}(b) = \frac{m}{n},$$

   and multiplying both sides of the equation by $\frac{l}{k}$ gives

$$b = \frac{m}{n}\frac{l}{k} = \frac{ml}{nk}.$$

   Since ml and nk are integers and $nk \neq 0$, we have that b is rational, a contradiction. □

3. If $a \equiv b \pmod{n}$, then $\gcd(a, n) = \gcd(b, n)$.

   Solution:

   Proof. (direct) Suppose that $a \equiv b \pmod{n}$. Then, $n \mid (a - b)$, so there exists an integer n such that $a - b = kn$. Let $d = \gcd(a, n)$. Then $d \mid a$ and $d \mid n$, so there exist integers x and y such that $a = dx$ and $n = dy$. Substituting this into $a - b = kn$ gives $dx - b = k(dy)$, or

   $$b = dx - dky = d(x - ky),$$

   so since $x - ky \in \mathbb{Z}$, $d \mid b$. Since $d \mid n$ and $d \mid b$, $\gcd(b, n) \geq d$, or $\gcd(b, n) \geq \gcd(a, n)$.

   Now, let $e = \gcd(b, n)$. Then, $e \mid b$ and $e \mid n$, so there exist integers w and z such that $b = ew$ and $n = ez$. Substituting this into $a - b = kn$ gives $a - ew = k(ez)$, or

   $$a = ekz + ew = e(kz + w),$$

   and so, since $kz + w \in z$, $d \mid a$. Since $e \mid a$ and $e \mid n$, $e \leq \gcd(a, n)$, or $\gcd(b, n) \leq \gcd(a, n)$.

   Since we have $\gcd(a, n) \leq \gcd(b, n)$ and $\gcd(a, n) \geq \gcd(b, n)$, it must be true that $\gcd(a, n) = \gcd(b, n)$. $\square$

4. Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

   Solution:

   Proof. (contrapositive) Suppose that a is an even integer. Then $a = 2k$ for some integer k. Therefore, $a^2 = (2k)^2 = 4k^2$, so since k^2 is an integer, a^2 is divisible by 4. $\square$

5. If $a \in \mathbb{Z}$ and $a \equiv 1 \pmod{5}$, then $a^2 \equiv 1 \pmod{5}$.

   Solution:

   Proof. (direct)

   Let $a \in \mathbb{Z}$ and $a \equiv 1 \pmod{5}$. Then $5 \mid (a - 1)$, so there exists an integer n such that $a - 1 = 5n$. Therefore,

   $$\begin{aligned} a &= 5n + 1 \\ a^2 &= (5n + 1)^2 \\ &= 25n^2 + 10n + 1 \\ &= 5(5n^2 + 2n) + 1. \end{aligned}$$

   So,

   $$a^2 - 1 = 5(n^2 + 2n).$$

   Since $n^2 + 2n$ is an integer, $5 \mid (a^2 - 1)$, or $a^2 \equiv 1 \pmod{5}$. $\square$

6. If a and b are positive real numbers, then $a+b \geq 2\sqrt{ab}$.

Solution: Side work: I'm going to do some algebra to see if something comes to mind.

$$\begin{aligned} a+b &\geq 2\sqrt{ab} \\ (a+b)^2 &\geq (2\sqrt{ab})^2 \\ a^2+2ab+b^2 &\geq 4ab \\ a^2-2ab+b^2 &\geq 0 \\ (a-b)^2 &\geq 0 \end{aligned}$$

So, two approaches to this proof come to mind: direct or by contradiction.

Proof. (direct) Let a and b be positive real numbers. Then $(a-b)^2 \geq 0$. Therefore,

$$a^2-2ab+b^2 \geq 0.$$

Adding $4ab$ to both sides gives

$$\begin{aligned} a^2+2ab+b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab. \end{aligned}$$

Since a, b, and $(a+b)^2$ are all positive, we can take the square root of both sides, obtaining

$$a+b \geq \sqrt{4ab} = 2\sqrt{ab}.$$

Therefore, $a+b \geq 2\sqrt{ab}$ for any positive real numbers a and b. $\square$

Proof. (contradiction) Suppose to the contrary that a and b are positive real numbers such that $a+b < 2\sqrt{ab}$. Then, since $(a+b)^2$ and $2\sqrt{ab}$ are nonnegative, we can take the square of both sides, and we have

$$\begin{aligned} (a+b)^2 &< [2\sqrt{ab}]^2 \\ a^2+2ab+b^2 &< 4ab \\ a^2-2ab+b^2 &< 0 \\ (a-b)^2 &< 0, \end{aligned}$$

a contradiction. Therefore, $a+b \geq 2\sqrt{ab}$ for any positive real numbers a and b. $\square$

7. Let $a \in \mathbb{Z}$. If $(a+1)^2 - 1$ is even, then a is even.

   Solution:

   Proof. (contrapositive) Suppose that a is an odd integer. Then $a = 2k+1$ for some integer k. So

$$\begin{aligned}(a+1)^2 - 1 &= (2k+2)^2 - 1\\ &= 4k^2 + 8k + 3\\ &= 4k^2 + 8k + 2 + 1\\ &= 2(2k^2 + 4k + 1) + 1.\end{aligned}$$

   Since $2k^2 + 4k + 1$ is an integer, $(a+1)^2 - 1$ is odd. □

8. Let $a, b \in \mathbb{Z}$. If $a \geq 2$, then either $a \nmid b$ or $a \nmid (b+1)$.

   Solution:

   Proof. (contradiction) Suppose to the contrary, that there exist integers a and b such that $a \geq 2$ and both $a \mid b$ and $a \mid (b+1)$. Since $a \mid b$, then $b = ax$ for some integer x. Since $a \mid (b+1)$, then $b+1 = ay$ for some integer y. Solving for b gives $b = ay - 1$. Equating the two expressions gives $ax = ay - 1$, or $ay - ax = 1$, which gives

$$a(y-x) = 1.$$

   Since a and $y - x$ are integers and $a \geq 2$, this is a contradiction. □

9. Evaluate the proof of the following proposition.

   Proposition. Let $n \in \mathbb{Z}$. If $3n - 8$ is odd, then n is odd.

   Proof. Assume that n is odd. Then $n = 2k+1$ for some integer k. Then

$$3n - 8 = 3(2k+1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k-3) + 1.$$

   Since $3k - 3$ is an integer, $3n - 8$ is odd. □

   Solution: It appears that the person writing the proof tried to do a proof by contrapositive. However, what the proof really shows is that if n is an odd integer, then $3n - 8$ is odd, the converse of the proposition. To prove the given proposition, we would use proof by contrapositive in which we would prove that if n is an even integer, then $3n - 8$ is even.