Homework # 9 Solutions

Math 111, Fall 2014
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Determine whether or not each of the statements is true or false. Prove your assertion.

1. Suppose $A, B,$ and C are sets. If $A \subseteq B$, then $A - C \subseteq B - C$.

   Solution: This statement is *true*.

   Proof. It is straightforward to show that if any of A, B, or C is the empty set, then the statement is true. So, let $A, B,$ and C be nonempty sets such that $A \subseteq B$. Then $x \in A - C$ means that $x \in A$ and $x \notin C$. Since $A \subseteq B$, if $x \in A$, then $x \in B$. So, $x \in B$ and $x \notin C$, or $x \in B - C$. Therefore, $A - C \subseteq B - C$. □

2. If $A, B,$ and C are sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   Solution: This statement is *true*.

   Proof. It is straightforward to show that if any of A, B, or C is the empty set, then the statement is true. Suppose, then, that $A, B,$ and C are nonempty sets. Then $z \in A \times (B \cup C)$ means that $z = (x, y)$, where $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, it follows that $y \in B$ or $y \in C$. So, we have that $x \in A$ and $y \in B$ or $x \in A$ and $y \in B$. Therefore, $z \in A \times B$ or $z \in A \times C$, from which it follows that $z \in (A \times B) \cup (A \times C)$. So, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Conversely, if we assume that $w \in (A \times B) \cup (A \times C)$, we have that $w \in A \times B$ or $w \in A \times C$. It follows that $w = (x, y)$, where $x \in A$ and $y \in B$ or $x \in A$ and $y \in B$. This means that $x \in A$ and $y \in B$ or $y \in C$. So, $w \in A \times (B \cup C)$, and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □

   As another proof, we can use the definitions given in Chapter 8.

   Proof.

$$
\begin{aligned}
A \times (B \cup C) &= \{(x, y) : (x \in A) \wedge (y \in B \cup C)\} \quad \text{(definition of Cartesian product)} \\
&= \{(x, y) : (x \in A) \wedge ((y \in B) \vee (y \in C))\} \quad \text{(definition of union)} \\
&= \{(x, y) : ((x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C))\} \quad \text{(distributive law)} \\
&= \{(x, y) : (x \in A) \wedge (y \in B)\} \cup \{(x, y) : (x \in A) \wedge (y \in C)\} \quad \text{(definition of union)} \\
&= (A \times B) \cup (A \times C) \quad \text{(definition of Cartesian product)}
\end{aligned}
$$

□

3. Suppose that A and B are sets. Then $A \subseteq B$ if and only if $A \cap B = A$.

Solution: This statement is *true*.

Proof. It is straightforward to prove that the statement is true if A or B (or both) is the empty set. So, we will assum that both are nonempty.

$\underline{\Longrightarrow|}$ Suppose that A and B are nonempty sets and $A \subseteq B$. Since $A \subseteq B$, if $x \in A$ then we also have $x \in B$. Therefore, $x \in A$ and $x \in B$, or $x \in A \cap B$. Since this is true for every $x \in A$, we have $A \subseteq A \cap B$. Similarly, if $x \in A \cap B$, then $x \in A$ and $x \in B$, so $A \cap B \subseteq A$. Therefore, $A \cap B = A$.

$\underline{\Longleftarrow|}$ Suppose that A and B are nonempty sets such that $A \cap B = A$. If $A \cap B = A$, then $A \subseteq A \cap B$, so if $x \in A$, then $x \in A$ and $x \in B$. So, we have that if $x \in A$, then $x \in B$, or $A \subseteq B$.

$\square$

4. For every rational number $\frac{a}{b}$, where $a, b \in \mathbb{N}$, there exists a rational number $\frac{c}{d}$, where c and d are positive odd integers, such that $0 < \frac{c}{d} < \frac{a}{b}$.

Solution: This statement is *true*. Here are two possible proofs of this statement.

Proof. Let $x = \frac{a}{b}$, where $a, b \in \mathbb{N}$. Without loss of generality, assume that $\gcd(a, b) = 1$ (i.e., the fraction is in reduced form). Then, let $c = 1$ and $d = 2b+1$. Clearly, c and d are both positive odd integers. It remains for us to show that $\frac{c}{d} < \frac{a}{b}$. There are two cases to consider: $a = 1$ and $a > 1$. If $a = 1$, then $c = a$. Since $d = 2b + 1 > b$, it follows that $\frac{c}{d} < \frac{a}{b}$. If $a > 1$, then $c < a$ and $d > b$; therefore, $\frac{c}{d} < \frac{a}{b}$. Therefore, for every rational number $\frac{a}{b}$, where $a, b \in \mathbb{N}$, there exists a rational number $\frac{c}{d}$, where c and d are positive odd integers such that $0 < \frac{c}{d} < \frac{a}{b}$. $\square$

Proof. Let $x = \frac{a}{b}$, where $a, b \in \mathbb{N}$. Without loss of generality, assume that $\gcd(a, b) = 1$ (i.e., the fraction is in reduced form). Then, we have two cases: b is even and b is odd.

Case 1: The denominator b is even.

In this case, we know that a is odd (since $\gcd(a, b) = 1$), so define $c = a$ and $d = b + 1$. Then $0 < \frac{c}{d} < \frac{a}{b}$.

Case 2: The denominator b is odd.

In this case, if a is odd, then we simply define $c = 1$ and $d = b + 2$. If a is even, then let $c = a - 1$ and let $d = b + 2$. We know that $a - 1 > 0$, since $a \geq 2$ if a is even. In both cases, we have $0 < \frac{a}{b} < \frac{c}{d}$.

Therefore, for every rational number $\frac{a}{b}$, where $a, b \in \mathbb{N}$, there exists a rational number $\frac{c}{d}$, where c and d are positive odd integers, such that $0 < \frac{c}{d} < \frac{a}{b}$. $\square$

5. If A and B are sets and $A \cap B = \varnothing$, then $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   Solution: This statement is *true.*

   Proof. Let $X \in \mathcal{P}(A) - \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ and $X \notin \mathcal{P}(B)$. Since $A \cap B = \varnothing$, it follows that $A - B = A$ (since $A - B = \{x : (x \in A) \wedge (x \notin B)\}$ and $A \cap B = \varnothing$ means that if $x \in A$, $x \notin B$). Therefore, $\mathcal{P}(A - B) = \mathcal{P}(A)$. Since $X \in \mathcal{P}(A)$ and $\mathcal{P}(A) = \mathcal{P}(A - B)$, it follows that $X \in \mathcal{P}(A - B$. Therefore, since $X \in \mathcal{P}(A) - \mathcal{P}(B)$ means that $X \in \mathcal{P}(A)$ which implies $X \in \mathcal{P}(A - B)$, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$. $\square$

6. For all positive real numbers x, $2^x \geq x + 1$.

   Solution: This statement is *false.*

   Counterexample. Let $x = \frac{1}{2}$. Then $2^x = \sqrt{2} \approx 1.414$ and $x + 1 = \frac{1}{2} + 1 = \frac{3}{2} > 2^{\frac{1}{2}}$. So, we have found a value of x for which $2^x < x + 1$.

7. Suppose $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \mid a$, then $a = b$.

   Solution: This statement is *false.*

   Counterexample. Let $a = -2$ and $b = 2$. Then $a \mid b$ since $-2 = (-1)2$ and $b \mid a$ since $2 = (-1)(-2)$, but $a \neq b$.

8. The sum of two distinct irrational numbers is irrational.

   Solution: This statement is *false.*

   Counterexample. Consider the irrational numbers $\sqrt{2}$ and $-\sqrt{2}$. We have that $\sqrt{2} + (-\sqrt{2}) = 0$, an integer.

9. If x and y are real numbers such that $|x + y| = |x| + |y|$, then either $x = 0$ or $y = 0$.

   Solution: This statement is *false.*

   Counterexample. Let $x = 2$ and $y = 3$. Then $|x + y| = |2 + 3| = |5| = 5$ and $|x| + |y| = |2| + |3| = 2 + 3 = 5$, so $|x + y| = |x| + |y|$, but $x \neq 0$ and $y \neq 0$.