# Functions - Chapter 12 of Hammack 

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### 12.1 Functions

You have seen functions in all of your math classes, and hopefully each time learned a bit more about them. So, let's consider a familiar function from Calculus, $f(x)=e^{x}$. The graph of $f(x)$ is the set $G=\left\{\left(x, e^{x}\right): x \in \mathbb{R}\right\} \subseteq \mathbb{R} \times \mathbb{R}$. Since $G \subseteq \mathbb{R} \times \mathbb{R}$, from our work in the previous chapter, we see that $G$ is a relation on the set $\mathbb{R}$.

Let's consider another example, $g(n)=|n|+1$, where $n \in \mathbb{Z}$. Then the domain of $g$ is all of $\mathbb{Z}$. The function $g$ takes integer values that are greater than or equal to 1 , so the range of $g$ is $\mathbb{N}$. So, the graph of $g$ is $R=\{(n,|n|+1): n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{N}$. In other words, the graph of $g$ is a relation from $\mathbb{Z}$ to $\mathbb{N}$.

Definition. Suppose $A$ and $B$ are sets. A function $f$ from $A$ to $B$ (denoted as $f: A \rightarrow B$ ) is a relation from $A$ to $B$, saisfying the property that for each $a \in A$, the relation $f \subseteq A \times B$ contains exactly one ordered pair of the form $(a, b)$. The statement $(a, b) \in f$ is abbreviated $f(a)=b$.

Consider one of the functions we discussed earlier, $f=\left\{\left(x, e^{x}\right): x \in \mathbb{R}\right\} \subseteq \mathbb{R} \times \mathbb{R}$. For every $a \in \mathbb{R}$, the set $f$ contains exactly one ordered pair $\left(a, e^{a}\right)$ whose first coordinate is $a$. For example, $(0,1) \in f$, so we write $f(0)=1$, and $\left(-1, e^{-1}\right) \in f$, so we write $f(-1)=e^{-1}$.

In general, $(a, b) \in f$ means that $f$ sends the input value $a$ to the output value $b$, and we express this as $f(a)=b$.
Definition. For a function $f: A \rightarrow B$, the set $A$ is called the domain of $f$. The set $B$ is called the codomain of $f$. The range of $f$ is the set $\{f(a): a \in A\}=\{b:(a, b) \in f\}$.

You can think of the domain as the set of all possible "input values" and the range as the set of all possible "output values." The codomain is like a "target" for the output.

## Example:

- Consider the function $f: \mathbb{Z} \rightarrow \mathbb{N}$ defined by $f(n)=|n|+1$. Then the domain of $f$ is $\mathbb{Z}$, and the range of $f$ is $\{f(a): a \in \mathbb{Z}\}=\{1,2,3,4, \ldots\}=\mathbb{N}$. In this case, the range is equal to the codomain. This is not always the case.
- Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=\cos x$. Then the domain of $f$ is $\mathbb{R}$, and the range of $f$ is $[-1,1]$. In this case, the range is a subset of the codomain $(\mathbb{R})$.


## Exercise:

(1) Suppose $A=\{0,1,2,3,4\}, B=\{2,3,4,5\}$, and $f=\{(0,3),(1,3),(2,4),(3,2),(4,3)\}$. State the domain and range of $f$. What is $f(2) ? f(1)$ ?

Solution: The domain of $f$ is $A$. The range of $f$ is $R=\{2,3,4\}$. The value of $f(2)$ is 4 , and the value of $f(1)$ is 1 .
(2) Consider the function $f: \mathbb{R}-\{1\} \rightarrow \mathbb{R}$ defined by $f(x)=\frac{1}{x-1}$. What is the domain, codomain, and range of $f$ ?

Solution: The domain of $f$ is $\mathbb{R}-\{1\}$; the codomain of $f$ is $\mathbb{R}$; and the range of $f$ is $\mathbb{R}-\{0\}$.

Let $A=\{1,2,3,4\}$ and let $B=\{x, y, z\}$. Then $f_{1}=\{(1, y),(2, x),(3, z),(4, x)\}$ is a function from $A$ to $B$, and so we may write $f_{1}: A \rightarrow B$. However, $f_{2}=\{(1, x),(2, z),(3, y),(2, x),(4, y)\}$ is not a function from $A \rightarrow B$, because there are two ordered pairs whose first coordinate is 2 . In addition, $f_{3}=\{(1, z),(3, x),(4, y)\}$ is not a function from $A$ to $B$, because the domain of $f_{3}$ is not all of $A$. The function $f_{3}$ is a function from $A-\{2\}$ to $B$.

It is often convenient to "visualize" a function $f: A \rightarrow B$ by representing the two sets $A$ and $B$ by diagrams and drawing an arrow from an element $a \in A$ to its image $f(a) \in B$. An example of this is below.


If $A$ and $B$ are infinite sets of numbers, such as $\mathbb{R}, \mathbb{Z}$, etc., then a graph in the traditional sense is the best way to visualize a function. For functions where $A$ and $B$ are finite sets, graphs as we did in the above example may be more useful. This is also true if we are thinking of $A$ and $B$ as generic sets.

For functions whose domain is a Cartesian product, such as $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$, then the function takes as input an ordered pair $(x, y) \in \mathbb{R}^{2}$ and sends it to a number $f((x, y)) \in \mathbb{R}$. In this case, the standard notation used is to write $f(x, y)$.

Suppose that $f: A \rightarrow B$ and $g: A \rightarrow B$ are two functions from $A$ to $B$ and let $a \in A$. Then $f$ and $g$ contain exactly one ordered pair having $a$ as its first coordinate. Suppose that $(a, x) \in f$ and $(a, y) \in g$. If the sets $f$ and $g$ are equal, then $(a, x) \in g$ as well. Since $g$ only contains one ordered pair whose first coordinate is $a$, it follows that $(a, x)=(a, y)$, or $x=y$, and so $f(a)=g(a)$. So, it is natural to define two functions as being equal as follows.
Definition. Two functions $f: A \rightarrow B$ and $g: A \rightarrow B$ are equal, written $f=g$, if $f(a)=g(a)$ for all $a \in A$.

We can generalize this a bit more to the following.
Definition. Two functions $f: A \rightarrow B$ and $g: C \rightarrow D$ are equal, written $f=g$, if $A=C, B=$ $D$, and $f(a)=g(a)$ for all $a \in A$.

So, to show that two functions $f$ and $g$ are equal, we need to confirm that their domains are equal, their codomains are equal, and that $f(a)=g(a)$ for every $a$ in the domain.

### 12.2 Injective and Surjective Functions

Recall from calculus that a function may be one-to-one or onto, and that these properties determine if a function is invertible. In advanced mathematics, these properties go by the names injective and surjective, respectively.

Definition. A function $f: A \rightarrow B$ is:

1. injective (or one-to-one) if for every $x, y \in A, x \neq y$ implies $f(x) \neq f(y)$;
2. surjective (or onto) if for every $b \in B$, there is an $a \in A$ such that $f(a)=b$;
3. bijective if $f$ is both injective and surjective.

Example: The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=x^{2}-3 x-2$ is not one-to-one, because $f(0)=-2$ and $f(3)=-2$.

As this example indicates, to show that a function is not injective only requires finding $x, y \in A$ for which $x \neq y$ and $f(x)=f(y)$.

There are two main approaches for proving that a function is injective: direct proof (assume $x, y \in A$ and $x \neq y$ and showing $f(x) \neq f(y))$ or proof by contrapositive (assume $x, y \in A$ and $f(x)=f(y)$, and showing that $x=y)$. Proof by contrapositive is frequently the easiest to use.

Example: Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=3 x-5$. Prove that $f$ is injective.

Proof. (Contrapositive.) Let $x, y \in A$ such that $f(x)=f(y)$. Since $f(x)=3 x-5$ and $f(y)=3 y-5$, we have that $3 x-5=3 y-5$. Adding 5 to both sides and dividing by 3 gives $x=y$. Therefore, $f$ is injective.

To prove that a function is surjective is typically done via a direct proof in which we assume that $b \in B$ and show that there exists a point $a \in A$ for which $f(a)=b$.

Example: Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=3 x-5$. Show that $f$ is surjectve.
Side work: If $y \in \mathbb{R}$ is such that $f(x)=y$, then $y=3 x-5$. Solving for $x$ gives $x=\frac{y+5}{3}$.
Proof. Let $y \in \mathbb{R}$. We show that there exists $x \in \mathbb{R}$ such that $f(x)=y$. Let $x=\frac{y+5}{3}$. Then $f\left(\frac{y+5}{3}\right)=3\left(\frac{y+5}{3}\right)-5=y$. Therefore, $f$ is surjective.

Exercise: Show that the function $f: \mathbb{R}-\{1\} \rightarrow \mathbb{R}-\{0\}$ defined by $f(x)=\frac{1}{x-1}$ is surjective.
Solution: Side work: Let $y \in \mathbb{R}$ such that $y=\frac{1}{x-1}$. Then

$$
\begin{aligned}
y & =\frac{1}{x-1} \\
y(x-1) & =1 \\
x-1 & =\frac{1}{y} \\
x & =\frac{1}{y}+1 .
\end{aligned}
$$

Proof. Let $y \in \mathbb{R}-\{0\}$. We show that there exists $x \in \mathbb{R}-\{1\}$ such that $f(x)=y$. Let $x=\frac{1}{y}+1$. Then

$$
\begin{aligned}
f(x) & =\frac{1}{\frac{1}{y}+1-1} \\
& =\frac{1}{\frac{1}{y}} \\
& =y .
\end{aligned}
$$

Therefore, $f$ is surjective.

To show that a function is bijective, we simply show that it is both injective and surjective.

### 12.3 The Pigeonhole Principle

The idea of the pigeonhole principle is the following. Suppose that we have a set $A$ of pigeons and a set $B$ of pigeon-holes, and all of the pigeons want to fly into the pigeon-holes. If there are more pigons than holes, then more than two pigeons will have to fly into at least one hole. If there are more holes than pigeons, then each pigeon can find a hole, but some holes will be empty. If we think of this as describing a function $f: A \rightarrow B$, where pigeon $X$ flies into pigeon-hole $f(X)$. If there are more pigeons than holes (i.e., if $|A|>|B|$ ), then for some $X \in A$, since more than one pigeon has to fly into at least one hole, $f$ is not injective. If there are fewer pigeons than pigeon-holes (i.e., if $|A|<|B|$ ), then at least one hole remains empty, and so $f$ is not surjective.

Theorem 1 (The Pigeonhole Principle). Suppose $A$ and $B$ are finite sets and $f: A \rightarrow B$ is any function. Then the following hold.

1. If $|A|>|B|$, then $f$ is not injective.
2. If $|A|<|B|$, then $f$ is not surjective.

Example: Prove that if six numbers are chosen at random, then at least two of them will have the same remainder when divided by 5 .

Solution: To prove this proposition, we need to use the pigeonhole principle.

Proof. We can make a partition of $\mathbb{Z}$ into 5 sets as follows: let $\mathbb{Z}=\bigcup_{j=0}^{4}\{5 k+j: k \in \mathbb{Z}\}$. If six integers are picked at random, by the pigeonhole principle, at least two will be in the same set. But, each set corresponds to the remainder of a number after being divided by 5 .

### 12.4 Composition

The notion of the composition of two functions was discussed in algebra, and again reviewed in calculus.

Definition. Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$ are functions with the property that the codomain of $f$ equals the domain of $g$. The composition of $f$ with $g$ is another function, denoted as $g \circ f$ and defined as follows: If $x \in A$, then $(g \circ f)(x)=g(f(x))$. Therefore, $g \circ f$ sends elements of $A$ to elements of $C$, so $g \circ f: A \rightarrow C$.

We can denote this schematically as shown in Figure 1.
Note: The composition $g \circ f$ also makes sense if the range of $f$ is a subset of the domain of $g$. To simplify things, we will mainly explore situations where the range of $f$ is equal to the domain of $g$.

Figure 1: Composition of two functions.


Example: Let $A=\{1,2,3,4\}, B=\{a, b, c, d\}$, and $C=\{r, s, t, u, v\}$ and define the functions $f: A \rightarrow B$ and $g: B \rightarrow C$ by

$$
\begin{aligned}
f & =\{(1, b),(2, d),(3, a),(4, a)\} \\
g & =\{(a, u),(b, r),(c, r),(d, s)\} .
\end{aligned}
$$

Determine $g \circ f$.
Solution: We have that $g \circ f$ is a function from $A$ to $C$. Since $f(1)=b$ and $g(b)=r$, we have that $g(f(1))=r ; f(2)=d$ and $g(d)=s$, so $g(f(2))=s ; f(3)=a$ and $g(a)=u$, so $g(f(3))=u$; and $f(4)=a$, so $g(f(4))=u$. Therefore, $g \circ f$ is defined as

$$
g \circ f=\{(1, r),(2, s),(3, u),(4, u)\}
$$

Exercise: Suppose $A=\{1,2,3\}$. Let $f: A \rightarrow A$ be the function $f=\{(1,2),(2,2),(3,1)\}$, and let $g: A \rightarrow A$ be the function $g=\{(1,3),(2,1),(3,2)\}$. Find $g \circ f$ and $f \circ g$.

Solution: We first find $g \circ f$. We have $f(1)=2$ and $g(2)=1$, so $g(f(1))=1 ; f(2)=2$ and $g(2)=1$, so $g(f(2))=1$; and $f(3)=1$ and $g(1)=3$, so $g(f(3))=3$. Therefore, $g \circ f=\{(1,1),(2,1),(3,3)\}$.

Next, find $f \circ g$. We have $g(1)=3$ and $f(3)=1$, so $f(g(1))=1 ; g(2)=1$ and $f(1)=2$, so $f(g(2))=2$; and $g(3)=2$ and $f(2)=2$, so $f(g(3))=2$. Therefore, $f \circ g=\{(1,1),(2,2),(3,2)\}$.

The textbook proves that the composition of functions is associative; i.e., if $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$, then $(h \circ g) \circ f=h \circ(g \circ f)$. We will prove the following theorem.

Theorem 2. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.
(a) If $f$ and $g$ are injective, then so is $g \circ f$.
(b) If $f$ and $g$ are surjective, then so is $g \circ f$.

Proof. (a) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be injective functions. Assume that $(g \circ f)\left(a_{1}\right)=$ $(g \circ f)\left(a_{2}\right)$, where $a_{1}, a_{2} \in A$. By definition, $g\left(f\left(a_{1}\right)\right)=g\left(f\left(a_{2}\right)\right)$. Since $g$ is injective, it follows that $f\left(a_{1}\right)=f\left(a_{2}\right)$. Since $f$ is injective, it follows that $a_{1}=a_{2}$. Therefore, $g \circ f$ is injective.
(b) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be surjective functions and let $c \in C$. Since $g$ is surjective, there exists $b \in B$ such that $g(b)=c$. Since $f$ is surjective, there exists $a \in A$ such that $f(a)=b$. Therefore, $(g \circ f)(a)=g(f(a))=g(b)=c$, and so $g \circ f$ is also surjective.

Corollary 1. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijective functions, then $g \circ f$ is also bijective.

This corollary directly follows from parts (a) and (b) of the above theorem.
Exercise: Prove or disprove. There exist functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that $f$ is not surjective and $g \circ f$ is surjective.

Solution: This statement is true. Let $A=\{x \in \mathbb{R}: x \geq 0\}$. Define $f: A \rightarrow A$ by $f(x)=x^{2}$ and define $g: A \rightarrow A$ as $g(x)=\sqrt{x}$. Then $(g \circ f)(x)=\sqrt{x^{2}}=x$ (since $\left.x \geq 0\right)$. We need only show that $(g \circ f)$ is surjective. Let $y \in A$. Then $y$ is in the codomain of $g$ and in the domain of $f$, and since $(g \circ f)(y)=y$, we have shown that $(g \circ f)$ is surjective.

### 12.5 Inverse Functions

In calculus, we saw that if a function is injective and surjective, then it has an inverse function $f^{-1}$ such that $f^{-1}(f(x))=x$ for every $x$ in the domain of $f$.

Definition. Given a set $A$, the identity function on $A$ is the function $i_{A}: A \rightarrow A$ defined as $i_{A}(x)=x$ for every $x \in A$.

Example:
(1) Let $A=\{a, b, c\}$. Then $i_{A}=\{(a, a),(b, b),(c, c)\}$.
(2) For $\mathbb{Z}$, we have $i_{\mathbb{Z}}=\{(m, m): m \in \mathbb{Z}\}$.
(3) For $\mathbb{R}$, we have $i_{\mathbb{R}}=\{(x, x): x \in \mathbb{R}\}$.

Note: For any set $A$, the identity function $i_{A}$ is bijective. It is injective because if $i_{A}(x)=i_{A}(y)$ for some $x, y \in A$, then since $i_{A}(x)=x$ and $i_{A}(y)=y$, it follows that $x=y$. It is surjective because if we take any element $a$ in the codomain $A$, then $b$ is also in the domain $A$, and $i_{A}(a)=a$.

Definition. Given a relation $R$ from $A$ to $B$, the inverse relation of $R$ is the relation from $B$ to $A$ defined as $R^{-1}=\{(y, x):(x, y) \in R\}$. In other words, the inverse of $R$ is the relation $R^{-1}$ obtained by interchanging the elements in every ordered pair in $R$.

Example: Suppose that $A=\{a, b, c, d\}$ and $B=\{1,2,3\}$. Let

$$
R=\{(a, 1),(a, 3),(c, 2),(c, 3),(d, 1)\}
$$

be a relation from $A$ to $B$. Then

$$
R^{-1}=\{(1, a),(3, a),(2, c),(3, c),(1, d)\}
$$

is the inverse relation. In this case, neither $R$ nor $R^{-1}$ is a function.
Now, let's suppose that

$$
f=\{(a, 1),(b, 3),(d, 2),(c, 1)\}
$$

is a relation from $A$ to $B$. Then, we have that the inverse relation is

$$
f^{-1}=\{(1, a),(3, b),(2, d),(1, c)\} .
$$

In this case, $f$ is a function from $A$ to $B$, but $f^{-1}$ is not a function.
It may also be the case that both $f$ and its inverse are functions.
Q: When is it true that if $f$ is a function, then $f^{-1}$ is a function?
A: Suppose $f$ is a function from $A$ to $B$. If the relation $f^{-1}$ is a function from $B$ to $A$, then the domain of $f^{-1}$ is $B$. This implies that $f$ must be onto (surjective). If $f$ is not one-to-one (injective), then $f(x)=f(y)=b$ for some $x, y \in A$ where $x \neq y$. But then $(b, x),(b, y) \in f^{-1}$, which cannot occur if $f^{-1}$ is a function. This leasds us to the following theorem.

Theorem 3. Let $f: A \rightarrow B$ be a function. Then the inverse relation $f^{-1}$ is a function from $B$ to $A$ if and only if $f$ is bijective. Furthermore, if $f$ is bijective, then $f^{-1}$ is also bijective.

Proof. $\Rightarrow$ Assume that $f^{-1}$ is a function from $B$ to $A$. Then we need to show that $f$ is both injective and surjective. Assume that $f\left(a_{1}\right)=f\left(a_{2}\right)=b$, where $y \in B$. Then $\left(a_{1}, y\right),\left(a_{2}, y\right) \in f$, implying that $\left(y, a_{1}\right),\left(y, a_{2}\right) \in f^{-1}$. Since $f^{-1}$ is a function from $B$ to $A$, every element of $B$ has a unique image in $A$. In particular, $y$ has a unique image under $f^{-1}$. Since $f^{-1}(y)=a_{1}$ and $f^{-1}(y)=a_{2}$, it follows that $a_{1}=a_{2}$ and so $f$ is injective. To show that $f$ is surjective, let $b \in B$. Since $f^{-1}$ is a function from $B$ to $A$, there exists a unique element $a \in A$ such that $f^{-1}(b)=a$. Therefore, $(b, a) \in f^{-1}$, implying that $(a, b) \in f$, and so $f$ is onto.
$\Longleftarrow$ Assume the $f: A \rightarrow B$ is bijective. We show that $f^{-1}$ is a function from $B$ to $A$. Let $b \in B$. Since $f$ is onto, there exists $a \in A$ such that $(a, b) \in f$. Therefore, $(b, a) \in f^{-1}$. We need only show that $(b, a)$ is the unique element of $f^{-1}$ whose first coordinate is $b$. Suppose that $(b, a)$ and $\left(b, a^{\prime}\right)$ are both in $f^{-1}$. Then $(a, b),\left(a^{\prime}, b\right) \in f$, which implies that
$f(a)=f\left(a^{\prime}\right)=b$. Since $f$ is injective, $a=a^{\prime}$. Therefore, we have shown that for every $b \in B$, there exists a unique element $a \in A$ such that $(b, a) \in f^{-1}$. Therefore, $f^{-1}$ is a function from $B$ to $A$.

Finally, we show that if $f$ is bijective, then $f^{-1}$ is bijective. Assume that $f: A \rightarrow B$ is bijective. We have already proven that $f^{-1}$ is a function from $B$ to $A$. We need to show that $f^{-1}$ is injective and surjective. First, assume that $f^{-1}\left(b_{1}\right)=f^{-1}\left(b_{2}\right)=a$. Then $\left(b_{1}, a\right),\left(b_{2}, a\right) \in f^{-1}$ and so $\left(a, b_{1}\right),\left(a, b_{2}\right) \in f$. Since $f$ is a function, $b_{1}=b_{2}$ and $f^{-1}$ is injective. Next, let $a \in A$. Since $f$ is a function, there is a $b \in B$ such that $(a, b) \in f$. Consequently, $(b, a) \in f^{-1}$ so that $f^{-1}(b)=a$ and $f^{-1}$ is surjective. Since $f^{-1}$ is injective and surjective, by definition $f^{-1}$ is bijective.

Suppose that $f: A \rightarrow B$ is bijective. Then, according to the theorem, $f^{-1}$ is a bijective function. Note that $f$ contains all of the pairs $(x, f(x))$ for $x \in A$, so $f^{-1}$ contains all of the pairs $(f(x), x)$. But $(f(x), x) \in f^{-1}$ means that $f^{-1}(f(x))=x$. Therefore, $\left(f^{-1} \circ f\right)(x)=x$ for every $x \in A$. From this, we see that $f^{-1} \circ f=i_{A}$. Similarly, we can show that $f \circ f^{-1}=i_{B}$.
Definition. If $f: A \rightarrow B$ is bijective, then its inverse is the function $f^{-1}: B \rightarrow A$. Functions $f$ and $f^{-1}$ satisfy the following:

$$
f^{-1} \circ f=i_{A} \text { and } f \circ f^{-1}=i_{B}
$$

Example: The function $f: \mathbb{R}-\{2\} \rightarrow \mathbb{R}-\{3\}$ defined by

$$
f(x)=\frac{3 x}{x-2}
$$

is bijective. Determine $f^{-1}(x)$, where $x \in \mathbb{R}-\{3\}$.
Solution: Since $\left(f \circ f^{-1}\right)(x)=x$ for all $x \in \mathbb{R}-\{3\}$, we have that

$$
\left(f \circ f^{-1}\right)(x)=f\left(f^{-1}(x)\right)=\frac{3 f^{-1}(x)}{f^{-1}(x)-2}=x
$$

Let $y=f^{-1}(x)$ and solve for $y$. We have

$$
\begin{aligned}
x & =\frac{3 y}{y-2} \\
x(y-2) & =3 y \\
x y-2 x & =3 y \\
x y-3 y & =2 x \\
y(x-3) & =2 x \\
y & =\frac{2 x}{x-3} .
\end{aligned}
$$

Therefore,

$$
f^{-1}(x)=\frac{2 x}{x-3}
$$

Note: You can check your answer by verifying that $\left(f^{-1} \circ f\right)(x)=x$.
Example: The function $g: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ defind by the formula $g(m, n)=(m-n, m-2 n)$ is bijective. (Verify.) Find $g^{-1}$.

Solution: Since $\left(g \circ g^{-1}\right)(x, y)=(x, y)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. For all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, suppose $(x, y)=g(m, n)$, or $(m, n)=g^{-1}(x, y)$. Then switching variables gives $(x, y)=g^{-1}(m, n)$ and so we have

$$
\begin{aligned}
(m, n) & =(x-y, x-2 y) \\
\Longrightarrow m & =x-y \\
n & =x-2 y
\end{aligned}
$$

We must solve for $x$ and $y$. Subtracting the second equation from the first gives

$$
m-n=y,
$$

so

$$
\begin{aligned}
m & =x-(m-n) \\
x & =m+(m-n)=2 m-n .
\end{aligned}
$$

Therefore, we have that

$$
g^{-1}(m, n)=(2 m-n, m-n) .
$$

You should verify that $\left(g^{-1} \circ g\right)(m, n)=(m, n)$.
Exercise: Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=4 x-3$ is bijective and determine $f^{-1}(x)$ for $x \in \mathbb{R}$.

Solution: First, we show that $f$ is bijective. To do this, we show that (a) $f$ is injective, and (b) $f$ is surjective. Suppose that $f(x)=f(y)$ for $x, y \in \mathbb{R}$. Then $4 x-3=4 y-3$. Adding 3 to both sides and dividing by 4 gives $x=y$. Next, let $y \in \mathbb{R}$. Then, defining $x=\frac{y+3}{4}$ gives $f(x)=4\left(\frac{y+3}{4}\right)-3=y$. Therefore, $f$ is surjective. Since $f$ is both injective and surjective, $f$ is bijective.

Next, we need to find $f^{-1}$. Since $\left(f \circ f^{-1}\right)(x)=x$ for all $x \in \mathbb{R}$, we have that

$$
\left(f \circ f^{-1}\right)(x)=f\left(f^{-1}(x)\right)=4 f^{-1}(x)-3=x .
$$

Let $y=f^{-1}(x)$. Then, we have

$$
\begin{aligned}
x & =4 y-3 \\
4 y & =x+3 \\
y & =\frac{x+3}{4} .
\end{aligned}
$$

Therefore,

$$
f^{-1}(x)=\frac{x+3}{4}
$$

### 12.6 Image and Pre-Image

Suppose that we have a function $f: A \rightarrow B$. If $X \subseteq A$, the expression $f(X)=\{f(x): x \in X\}$. Similarly, if $Y \subseteq B$, then $f^{-1}(Y)=\{x \in A: f(x) \in Y\}$. Note that this is defined even if $f$ is not invertible. The precise definitions follow.

Definition. Suppose $f: A \rightarrow B$ is a function.

1. If $X \subseteq A$, the image of $X$ is the set $f(X)=\{f(x): x \in X\} \subseteq B$.
2. If $Y \subseteq B$, the pre-image of $Y$ is the set $f^{-1}(Y)=\{x \in A: f(x) \in Y\} \subseteq A$.

Example: Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x)=x^{2}+3$. Find $f([-3,5])$ and $f^{-1}([12,19])$.

Solution: We know that $f([-3,5])=\{f(x): x \in[-3,5]\}$. Since $f(x)=x^{2}+3$, we have that $f(-3)=(-3)^{2}+3=12$ and $f(5)=5^{2}+3=28$. Since $f^{\prime}(x)=2 x$, we know that $f$ is decreasing on $[-3,0]$ and increasing on $[0,5]$. Therefore, since $f(0)=3$, we have

$$
f([-3,5])=[3,28] .
$$

Next, we wish to find $f^{-1}([12,19])=\{x \in \mathbb{R}: f(x) \in[12,19]\}$. If $f(x)=12$, then $x=$ $\pm \sqrt{12-3}= \pm \sqrt{9}= \pm 3$. If $f(x)=19$, then $x= \pm \sqrt{19-3}= \pm \sqrt{16}= \pm 4$. If $x \in[-4,-3]$, then $f(x)$ is decreasing, and if $x \in[3,4]$, then $f$ is increasing (since $f^{\prime}(x)=2 x$ ). Therefore,

$$
f^{-1}([12,19])=[-4,-3] \cup[3,4] .
$$

Example: Let $f: A \rightarrow B$ be a function and $X \subseteq A$. Prove or disprove: $f\left(f^{-1}(f(X))\right)=f(X)$.
Solution: This is a true statement.

Proof. First, we prove that $f\left(f^{-1}(f(X))\right) \subseteq f(X)$. Suppose that $y \in f\left(f^{-1}(f(X))\right)$. By definition of image, this means that $y=f(x)$ for some $x \in f^{-1}(f(X))$. But, by definition of pre-image, $x \in f^{-1}(f(X))$ means that $f(x) \in f(X)$. Thus, we have $y=f(x) \in f(X)$ and $f\left(f^{-1}(f(X))\right) \subseteq f(X)$. Next, we show $f(X) \subseteq f\left(f^{-1}(f(X))\right)$. Suppose that $y \in f(X)$. This means that $y=f(x)$ for some $x \in X$. Then $f(x)=y \in f(X)$, which means $x \in f^{-1}(f(X))$. Then, by definition of image, $f(x) \in f\left(f^{-1}(f(X))\right)$. So, we have $y=f(x) \in f\left(f^{-1}(f(X))\right)$ and $f(X) \subseteq f\left(f^{-1}(f(X))\right)$. Therefore, $f\left(f^{-1}(f(X))\right)=f(X)$.

Exercise: Let $f: A \rightarrow B$ be a function and let $W, X \subseteq A$. Prove or disprove: $f(W \cap X) \subseteq$ $f(W) \cap f(X)$.

Solution: This is a true statement.

Proof. Let $b \in f(W \cap X)$. Then $b \in\{f(x): x \in W \cap X\}$. So, $b=f(a)$ for some $a \in W \cap X$. Therefore, $a \in W$ and $a \in X$. Since $a \in W$, we have $b=f(a) \in\{f(x): x \in W\}=f(W)$. Since $a \in X$, we have $b=f(a) \in\{f(x): x \in X\}=f(X)$. Therefore, $b$ is in both $f(W)$ and $f(X)$, or $b \in f(W) \cap f(X)$. Therefore, $f(W \cap X) \subseteq f(W) \cap f(X)$.

