Some Fourier Series Theorems
Math 182, Spring 2009
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- Convergence Theorems
  - Convergence of Fourier series: If $f(x)$, defined on the interval $[-L, L]$, is piecewise C^1 on $[-L, L]$, then the Fourier series of $f(x)$, $S[f](x)$, converges to:
    (i) $f(a)$ if $f(x)$ is continuous at $a \in (-L, L)$;
    (ii) $\frac{1}{2}(f(a-) + f(a+))$ if $f(x)$ has a jump discontinuity at $a \in (-L, L)$.
  - Convergence of Fourier sine series: Consider $f(x)$ defined on the interval $[0, L]$.
    - If $f(x)$ is piecewise C^1 on $[0, L]$, then the Fourier sine series of $f(x)$ converges to:
      (i) $f(a)$ if $f(x)$ is continuous at $a \in (0, L)$;
      (ii) $\frac{1}{2}(f(a-) + f(a+))$ if $f(x)$ has a jump discontinuity at $a \in (0, L)$.
    - If $f(x)$ is continuous and $f(0) = f(L) = 0$, then the Fourier sine series of $f(x)$ converges to $f(x)$ on $[0, L]$.
  - Convergence of Fourier cosine series: Consider $f(x)$ defined on the interval $[0, L]$.
    - If $f(x)$ is piecewise C^1 on $[0, L]$, then the Fourier sine series of $f(x)$ converges to:
      (i) $f(a)$ if $f(x)$ is continuous at $a \in (0, L)$;
      (ii) $\frac{1}{2}(f(a-) + f(a+))$ if $f(x)$ has a jump discontinuity at $a \in (0, L)$.
    - If $f(x)$ is continuous, then the Fourier cosine series of $f(x)$ converges to $f(x)$ on $[0, L]$.

- Term by Term Differentiation
  - Fourier series
    - Suppose $f(x)$, defined on the interval $[-L, L]$, is piecewise C^1 on $[-L, L]$ and the Fourier series of f, $S[f](x)$, is continuous (including the endpoints). Then its Fourier series can be differentiated term by term.
    - For the Fourier series of f to be continuous, it is sufficient to assume that $f(x)$ is continuous and $f(-L) = f(L)$.
  - Fourier cosine series: Suppose $f(x)$ is defined on the interval $[0, L]$. Then the Fourier cosine series of f can be differentiated term by term if f is continuous on $[0, L]$ and f' is piecewise C^1 on $[0, L]$.
  - Fourier sine series: Suppose $f(x)$ is defined on the interval $[0, L]$. Then the Fourier sine series of f can be differentiated term by term if f is continuous on $[0, L]$, f' is piecewise C^1 on $[0, L]$, and $f(0) = f(L) = 0$.

- Term by Term Integration
  - Fourier series: Suppose $f(x)$, defined on the interval $[-L, L]$, is piecewise C^1 on $[-L, L]$. Then the Fourier series of f can be integrated term by term and the obtained series is convergent to the integral of $f(x)$.