

Performing the indicated integration yields

$$\int_{-L}^x f(t) dt \sim a_0(x+L) + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi/L} \left( \cos n\pi - \cos \frac{n\pi x}{L} \right) \right]. \quad (3.5.2)$$

We will actually show that the preceding statement is valid with an = sign. If term-by-term integration from  $-L$  to  $x$  of a Fourier series is valid, then any definite integration is also valid since

$$\int_a^b = \int_{-L}^b - \int_{-L}^a.$$

**Example.** Term-by-term integration has some interesting applications. Recall that the Fourier sine series for  $f(x) = 1$  is given by

$$1 \sim \frac{4}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \dots \right), \quad (3.5.3)$$

where  $\sim$  is used since (3.5.3) is an equality only for  $0 < x < L$ . Integrating term by term from 0 to  $x$  results in

$$x \sim \frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \dots \right), \quad 0 \leq x \leq L, \quad (3.5.4)$$

where because of our theorem the = sign can be used. We immediately recognize that (3.5.4) should be the Fourier cosine series of the function  $x$ . It was obtained by integrating the Fourier sine series of  $f(x) = 1$ . However, an infinite series of constants appears in (3.5.4); it is the constant term of the Fourier cosine series of  $x$ . In this way we can evaluate that infinite series,

$$\frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = \frac{1}{L} \int_0^L x dx = \frac{1}{2}L.$$

Thus, we obtain the usual form for the Fourier cosine series for  $x$ ,

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \dots \right), \quad 0 \leq x \leq L. \quad (3.5.5)$$

The process of deriving new series from old ones can be continued. Integrating (3.5.5) from 0 to  $x$  yields

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3} \left( \sin \frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \dots \right). \quad (3.5.6)$$

This example illustrates that *integrating a Fourier series term by term does not necessarily yield another Fourier series*. However, (3.5.6) can be looked at as either yielding

1. The Fourier sine series of  $x^2/2 - (L/2)x$ , or
2. The Fourier sine series of  $x^2/2$ , where the Fourier sine series of  $x$  is needed first [see (3.3.11) and (3.3.12)].

An alternative procedure is to perform indefinite integration. In this case an arbitrary constant must be included and evaluated. For example, reconsider the Fourier sine series of  $f(x) = 1$ , (3.5.3). By term-by-term indefinite integration we derive the Fourier cosine series of  $x$ ,

$$x = c - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \dots \right).$$

The constant of integration is not arbitrary; it must be evaluated. Here  $c$  is again the constant term of the Fourier cosine series of  $x$ ,  $c = (1/L) \int_0^L x \, dx = L/2$ .

**Proof on integrating Fourier series.** Consider

$$F(x) = \int_{-L}^x f(t) \, dt. \quad (3.5.7)$$

This integral is a continuous function of  $x$  since  $f(x)$  is piecewise smooth.  $F(x)$  has a continuous Fourier series only if  $F(L) = F(-L)$  [otherwise, remember that the periodic nature of the Fourier series implies that the Fourier series does not converge to  $F(x)$  at the endpoints  $x = \pm L$ ]. However, note that from the definition (3.5.7),

$$F(-L) = 0 \quad \text{and} \quad F(L) = \int_{-L}^L f(t) \, dt = 2La_0.$$

Thus, in general  $F(x)$  does not have a continuous Fourier series. In Fig. 3.5.1,  $F(x)$  is sketched, illustrating the fact that usually  $F(-L) \neq F(L)$ . However, consider the straight line connecting the point  $F(-L)$  to  $F(L)$ ,  $y = a_0(x + L)$ .  $G(x)$ , defined to be the difference between  $F(x)$  and the straight line,

$$G(x) \equiv F(x) - a_0(x + L), \quad (3.5.8)$$

will be zero at both ends,  $x = \pm L$ ,

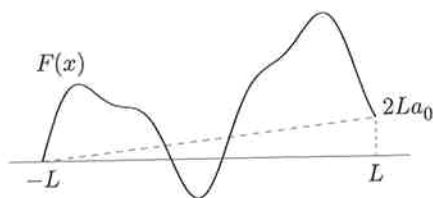
$$G(-L) = G(L) = 0,$$

as illustrated in Fig. 3.5.1.  $G(x)$  is also continuous. Thus,  $G(x)$  satisfies the properties that enable the Fourier series of  $G(x)$  actually to equal  $G(x)$ :

$$G(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right), \quad (3.5.9)$$

where the = sign is emphasized. These Fourier coefficients can be computed as

$$A_n = \frac{1}{L} \int_{-L}^L [F(x) - a_0(x + L)] \cos \frac{n\pi x}{L} \, dx \quad (n \neq 0).$$



**Figure 3.5.1**  $F(x)$  with  $F(-L) \neq F(L)$ .

The  $x$ -term can be dropped since it is odd (i.e.,  $\int_{-L}^L x \cos n\pi x/L \, dx = 0$ ). The resulting expression can be integrated by parts as follows:

$$\begin{aligned} u &= F(x) - a_0L & dv &= \cos \frac{n\pi x}{L} \, dx \\ du &= \frac{dF}{dx} \, dx = f(x) \, dx & v &= \frac{L}{n\pi} \sin \frac{n\pi x}{L}, \end{aligned}$$

yielding

$$A_n = \frac{1}{L} \left[ (F(x) - a_0L) \frac{\sin n\pi x/L}{n\pi/L} \Big|_{-L}^L - \frac{L}{n\pi} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} \, dx \right] = -\frac{b_n}{n\pi/L}, \quad (3.5.10)$$

where we have recognized that  $b_n$  is the Fourier sine coefficient of  $f(x)$ . In a similar manner (which we leave as an exercise), it can be shown that

$$B_n = \frac{a_n}{n\pi/L},$$

where  $a_n$  is the Fourier cosine coefficient of  $f(x)$ .  $A_0$  can be calculated in a different manner (the previous method will not work). Since  $G(L) = 0$  and the Fourier series of  $G(x)$  is pointwise convergent, from (3.5.9) it follows that

$$0 = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi = A_0 - \sum_{n=1}^{\infty} \frac{b_n}{n\pi/L} \cos n\pi$$

since  $A_n = -b_n/(n\pi/L)$ . Thus, we have shown from (3.5.9) that

$$F(x) = a_0(x+L) + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi/L} \left( \cos n\pi - \cos \frac{n\pi x}{L} \right) \right], \quad (3.5.11)$$

exactly the result of simple term-by-term integration. However, notice that (3.5.11) is *not* the Fourier series of  $F(x)$ , since  $a_0x$  appears. Nonetheless, (3.5.11) is valid. We have now justified term-by-term integration of Fourier series.

**EXERCISES 3.5**

3.5.1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \quad (3.5.12)$$

(a) Determine  $b_n$  from (3.3.11), (3.3.12), and (3.5.6).(b) For what values of  $x$  is (3.5.12) an equality?\*(c) Derive the Fourier cosine series for  $x^3$  from (3.5.12). For what values of  $x$  will this be an equality?3.5.2. (a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of  $x^2$ .(b) From part (a), determine the Fourier sine series of  $x^3$ .3.5.3. Generalize Exercise 3.5.2, in order to derive the Fourier sine series of  $x^m$ ,  $m$  odd.\*3.5.4. Suppose that  $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x/L$ .(a) Determine  $b_n$  by correctly differentiating this series twice.(b) Determine  $b_n$  by integrating this series twice.3.5.5. Show that  $B_n$  in (3.5.9) satisfies  $B_n = a_n/(n\pi/L)$ , where  $a_n$  is defined by (3.5.1).

3.5.6. Evaluate

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

by evaluating (3.5.5) at  $x = 0$ .

\*3.5.7. Evaluate

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

using (3.5.6).

**3.6 Complex Form of Fourier Series**

With periodic boundary conditions, we have found the theory of Fourier series to be quite useful:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \quad (3.6.1)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \quad (3.6.2)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad (3.6.3)$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \quad (3.6.4)$$

To introduce complex exponentials instead of sines and cosines, we use Euler's formulas

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

It follows that

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n) e^{in\pi x/L} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-in\pi x/L}. \quad (3.6.5)$$

In order to only have  $e^{-in\pi x/L}$ , we change the dummy index in the first summation, replacing  $n$  by  $-n$ . Thus,

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=-1}^{-\infty} [a_{(-n)} - ib_{(-n)}] e^{-in\pi x/L} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-in\pi x/L}.$$

From the definition of  $a_n$  and  $b_n$ , (3.6.3) and (3.6.4),  $a_{(-n)} = a_n$  and  $b_{(-n)} = -b_n$ . Thus, if we define

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{a_n + ib_n}{2}, \end{aligned}$$

then  $f(x)$  becomes simply

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}. \quad (3.6.6)$$

Equation (3.6.6) is known as the **complex form of the Fourier series of  $f(x)$** .<sup>3</sup> It is equivalent to the usual form. It is more compact to write, but it is only used infrequently. In this form the complex Fourier coefficients are

$$c_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{2L} \int_{-L}^L f(x) \left( \cos \frac{n\pi x}{L} + i \sin \frac{n\pi x}{L} \right) dx. \quad (n \neq 0)$$

<sup>3</sup>As before, an equal sign appears if  $f(x)$  is continuous [and periodic,  $f(-L) = f(L)$ ]. At a jump discontinuity of  $f(x)$  in the interior, the series converges to  $[f(x+) + f(x-)]/2$ .

We immediately recognize a simplification, using Euler's formula. Thus, we derive a formula for the complex Fourier coefficients

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{in\pi x/L} dx. \quad (\text{all } n) \quad (3.6.7)$$

Notice that the complex Fourier series representation of  $f(x)$  has  $e^{-in\pi x/L}$  and is summed over the discrete integers corresponding to the sum over the discrete eigenvalues. The complex Fourier coefficients, on the other hand, involve  $e^{+in\pi x/L}$  and are integrated over the region of definition of  $f(x)$  (with periodic boundary conditions), namely  $-L \leq x \leq L$ . If  $f(x)$  is real,  $c_{-n} = \bar{c}_n$  (see Exercise 3.6.2).

**Complex orthogonality**. There is an alternative way to derive the formula for the complex Fourier coefficients. Always, in the past, we have determined Fourier coefficients using the orthogonality of the eigenfunctions. A similar idea holds here. However, here the eigenfunctions  $e^{-in\pi x/L}$  are complex. For complex functions the concept of orthogonality must be slightly modified. A complex function  $\phi$  is said to be orthogonal to a complex function  $\psi$  (over an interval  $a \leq x \leq b$ ) if  $\int_a^b \bar{\phi} \psi dx = 0$ , where  $\bar{\phi}$  is the complex conjugate of  $\phi$ . This guarantees that the length squared of a complex function  $f$ , defined by  $\int_a^b \bar{f} f dx$ , is positive (this would not have been valid for  $\int_a^b f f dx$  since  $f$  is complex).

Using this notion of orthogonality, the eigenfunctions  $e^{-in\pi x/L}$ ,  $-\infty < n < \infty$ , can be verified to form an orthogonal set because by simple integration

$$\int_{-L}^L \overline{(e^{-im\pi x/L})} e^{-in\pi x/L} dx = \begin{cases} 0 & n \neq m \\ 2L & n = m, \end{cases}$$

since

$$\overline{(e^{-im\pi x/L})} = e^{im\pi x/L}.$$

Now to determine the complex Fourier coefficients  $c_n$ , we multiply (3.6.6) by  $e^{in\pi x/L}$  and integrate from  $-L$  to  $+L$  (assuming that the term-by-term use of these operations is valid). In this way

$$\int_{-L}^L f(x) e^{im\pi x/L} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^L e^{im\pi x/L} e^{-in\pi x/L} dx.$$

Using the orthogonality condition, the sum reduces to one term,  $n = m$ . Thus,

$$\int_{-L}^L f(x) e^{im\pi x/L} dx = 2L c_m,$$

which explains the  $1/2L$  in (3.6.7) as well as the switch of signs in the exponent.

**EXERCISES 3.6**

\*3.6.1. Consider

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta. \end{cases}$$

Assume that  $x_0 > -L$  and  $x_0 + \Delta < L$ . Determine the complex Fourier coefficients  $c_n$ .

3.6.2. If  $f(x)$  is real, show that  $c_{-n} = \bar{c}_n$ .