Homework #10 Solutions
Math 182, Spring 2009
Instructor: Dr. Doreen De Leon

[A] #1. Solve the PDE

$$\begin{cases} \rho_0 u_{tt} = T_0 u_{xx} + \alpha u, \ \rho_0 > 0, T_0 > 0, \alpha < 0. \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = f(x) \end{cases}$$

using separation of variables

Let $u(x,t) = \phi(x)T(t)$

$\Rightarrow u_{tt} = \phi(x)\dfrac{d^2T}{dt^2}, \quad u_{xx} = T(t)\dfrac{d^2\phi}{dx^2}, \quad \phi(0) = \phi(L) = 0$

$\Rightarrow \rho_0 \phi(x)\dfrac{d^2T}{dt^2} = T_0 T(t)\dfrac{d^2\phi}{dx^2} + \alpha\phi(x)T(t)$

Divide by $T_0\phi(x)T(t)$:

$$\frac{\rho_0}{T_0 T}\cdot\frac{d^2T}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} + \frac{\alpha}{T_0} = \text{constant} = -\lambda$$

$$\Rightarrow \frac{\rho_0}{T_0 T}\cdot\frac{d^2T}{dt^2} - \frac{\alpha}{T_0} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

Gives two equations:

A) $\rho_0 T'' - \alpha T + \lambda T_0 T = 0$ and

B) $\phi'' + \lambda\phi = 0$

Solving:

B) $\phi'' + \lambda\phi = 0, \ \phi(0) = \phi(L) = 0$

From our work with the heat equation:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n = 1, 2, \ldots \ ; \ \phi_n = \sin\left(\frac{n\pi}{L}x\right)$$

A) $\rho_0 T'' + (\lambda T_0 - \alpha)T = 0, \ T(0) = 0$

$\lambda > 0, \alpha < 0, \rho_0 > 0 \Rightarrow T_n(t) = c_1\cos\left(\sqrt{\frac{\lambda T_0 - \alpha}{\rho_0}}\, t\right) + c_2\sin\left(\sqrt{\frac{\lambda T_0 - \alpha}{\rho_0}}\, t\right)$

$\Rightarrow T_n(t) = c_1\cos\left(\sqrt{\frac{1}{\rho_0}\left(\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha\right)}\, t\right) + c_2\sin\left(\sqrt{\frac{1}{\rho_0}\left(\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha\right)}\, t\right)$

$T_n(0)=0 \Rightarrow Q = C_1$

So $T_n(t) = C \sin\left(\sqrt{\frac{1}{\rho_0}\left(\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha\right)}\ t\right)$

By superposition,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\frac{1}{\rho_0}\left(\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha\right)}\ t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\frac{du}{dt} = \sum_{n=1}^{\infty} \sqrt{\frac{1}{\rho_0}(\lambda T_0 - \alpha)}\ B_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\sqrt{\frac{1}{\rho_0}(\lambda T_0-\alpha)}\ t\right)$$

$$f(x) = u_t(x,0) = \sum_{n=1}^{\infty} \sqrt{\frac{1}{\rho_0}(\lambda T_0 - \alpha)}\ B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right) = \sum_{n=1}^{\infty} \sqrt{\frac{1}{\rho_0}(\lambda T_0-\alpha)}\ B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$= \sqrt{\frac{1}{\rho_0}(\lambda T_0 - \alpha)}\ B_m \cdot L/2$$

So $B_m = \frac{2}{L}\sqrt{\frac{\rho_0}{\lambda T_0 - \alpha}} \int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$

Solution:

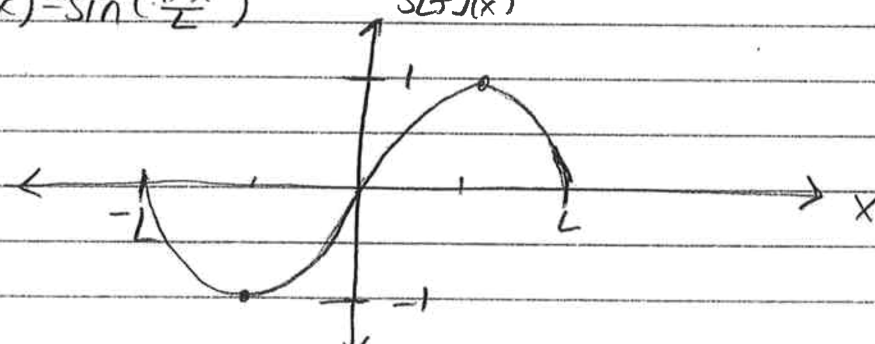
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\sqrt{\frac{1}{\rho_0}\left(\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha\right)}\ t\right)$$

where $B_n = \frac{2}{L}\sqrt{\frac{\rho_0}{\left(\frac{n\pi}{L}\right)^2 T_0 - \alpha}} \int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

[B] <u>Haberman: 3.2.2(c),(d),(e).</u>

For the following functions sketch the Fourier series of $f(x)$ on $-L \le x \le L$ and determine the Fourier coefficients

(c) $f(x) = \sin\left(\frac{\pi x}{L}\right)$



Fourier coefficients

$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$

$= \frac{1}{2L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)dx = \underline{0}$ (since $\sin\left(\frac{\pi x}{L}\right)$ is odd)

$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \underline{0}.$

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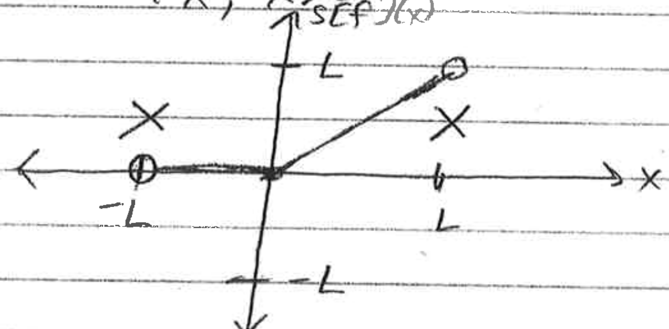
322(c)(cont) $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

$= \begin{cases} \frac{1}{L}(0) & n\neq 1 \\ \frac{1}{L}\cdot L = 1, & n=1 \end{cases}$

So $B_1=0$ and $A_0=A_n=0$, $B_n=0$ for $n\neq 1$.

* (d) $f(x)=\begin{cases} 0, & x<0 \\ x, & x>0 \end{cases}$



Fourier coefficients:

$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$

$= \frac{1}{2L}\int_0^L x\,dx$

$A_0 = \frac{1}{2L}\cdot\frac{x^2}{2}\Big|_0^L = \boxed{L/4 = A_0}$

$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\left[\frac{L}{n\pi}x\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right]$

$= -\frac{1}{n\pi}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$

$\boxed{A_n = \frac{L}{(n\pi)^2}(\cos(n\pi)-1)} = \begin{cases} -2L/(n\pi)^2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

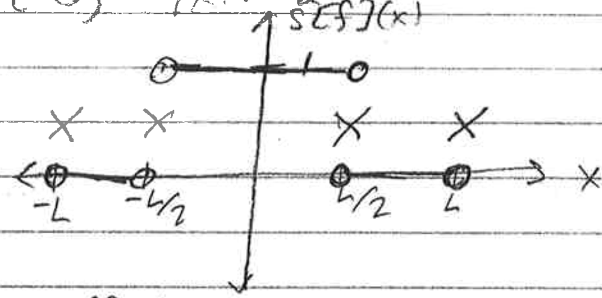
$= \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx\right]$

$= \frac{1}{L}\left[\frac{-L^2}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right]$

$\boxed{B_n = -\frac{L}{n\pi}\cos(n\pi)} = \begin{cases} L/n\pi, & n \text{ odd} \\ -L/n\pi, & n \text{ even.} \end{cases}$

e) $f(x) = \begin{cases} 1, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$



Fourier coefficients:

$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$
$= \frac{1}{2L}\int_{-L/2}^{L/2} 1\,dx = \boxed{\frac{1}{2} = A_0}$

$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$
$= \frac{1}{L}\int_{-L/2}^{L/2} \cos\left(\frac{n\pi x}{L}\right)dx$
$= \frac{1}{L}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L/2}^{L/2}$
$= \frac{1}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right)\right)$

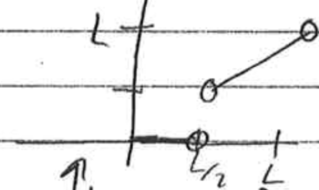
$$\boxed{A_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$
$= \frac{1}{L}\int_{-L/2}^{L/2} \sin\left(\frac{n\pi x}{L}\right)dx = \boxed{0 = B_n}$

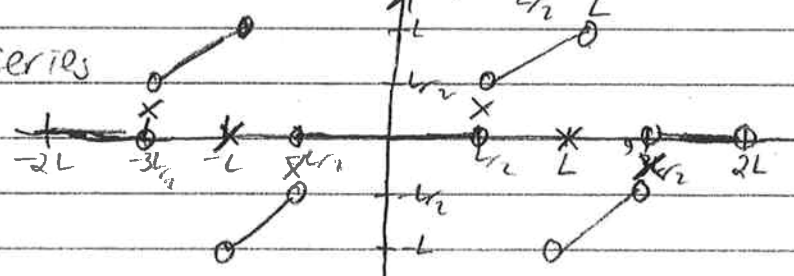
C 1. Haberman: 3.3.2 (c), (d)

For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients

* (c) $f(x) = \begin{cases} 0, & x < L/2 \\ x, & x > L/2 \end{cases}$



Fourier sine series



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3.3.2(c) (cont.)

Coefficients: $B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$

$$= \frac{2}{L}\int_{L/2}^{L} x\sin(\frac{n\pi}{L}x)dx$$

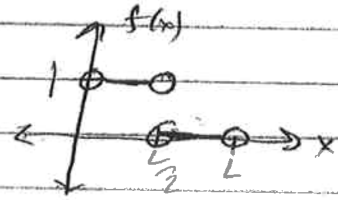
$$= \frac{2}{L}\left(-\frac{L}{n\pi}x\cos(\frac{n\pi}{L}x)\Big|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\cos(\frac{n\pi}{L}x)dx\right)$$

$$= \frac{2}{L}\left(-\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{2n\pi}\cos(\frac{n\pi}{2}) + \frac{L^2}{(n\pi)^2}\sin(\frac{n\pi}{L}x)\Big|_{L/2}^{L}\right)$$

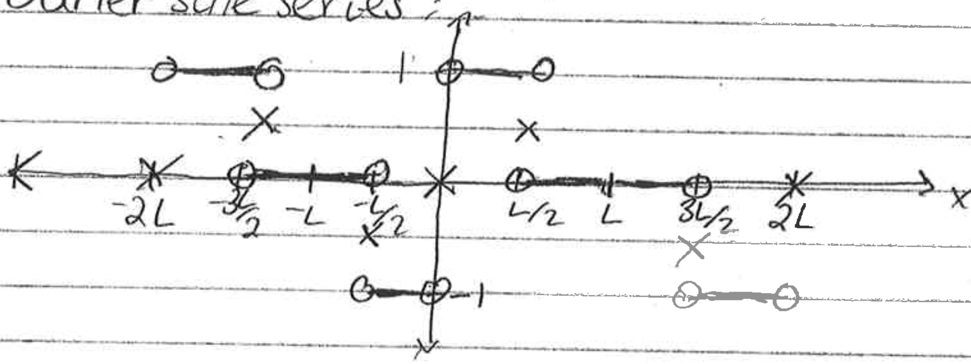
$$= \frac{2}{L}\left(-\frac{L^2}{n\pi}\cos(n\pi) + \frac{L^2}{2n\pi}\cos(\frac{n\pi}{2}) - (\frac{L}{n\pi})^2\sin(\frac{n\pi}{2})\right)$$

So $$\boxed{B_n = -\frac{2L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)}$$

(d) $f(x) = \begin{cases} 1, & x < L/2 \\ 0, & x > L/2 \end{cases}$



Fourier sine series:



Fourier coefficients

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

$$= \frac{2}{L}\int_0^{L/2}\sin(\frac{n\pi}{L}x)dx$$

$$= \frac{2}{L}\left(-\frac{L}{n\pi}\cos(\frac{n\pi}{L}x)\Big|_0^{L/2}\right)$$

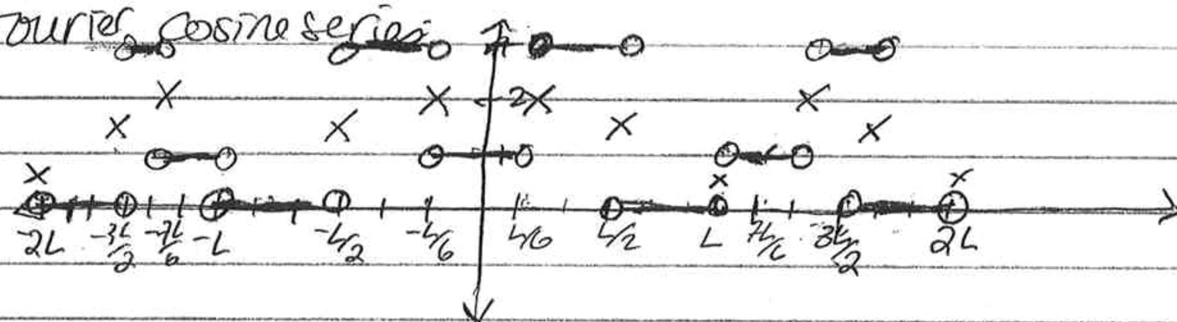
$$\boxed{B_n = \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)}$$

2. Haberman: 3.3.5(b),(c)

For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients

*b) $f(x) = \begin{cases} 1, & x < L/6 \\ 3, & L/6 < x < L/2 \\ 0, & x > L/2 \end{cases}$

Fourier cosine series:



Coefficients:

$$A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\left(\int_0^{L/6} dx + \int_{L/6}^{L/2} 3dx\right)$$

$$= \frac{1}{L}\left(\frac{L}{6} + 3x\Big|_{L/6}^{L/2}\right)$$

$$\boxed{A_0 = 7/6}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left(\int_0^{L/6}\cos\left(\frac{n\pi}{L}x\right)dx + \int_{L/6}^{L/2} 3\cos\left(\frac{n\pi}{L}x\right)dx\right)$$

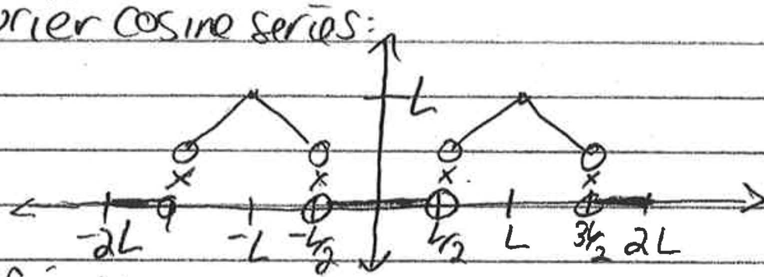
$$= \frac{2}{L}\left(\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{L/6} + \frac{3L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_{L/6}^{L/2}\right)$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\sin\left(\frac{n\pi}{6}\right) + \frac{3L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{3L}{n\pi}\sin\left(\frac{n\pi}{6}\right)\right)$$

$$\boxed{A_n = \frac{2}{n\pi}\left(3\sin\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{6}\right)\right)}$$

c) $f(x) = \begin{cases} 0, & x < L/2 \\ x, & x > L/2 \end{cases}$

Fourier cosine series:



Coefficients:

$$A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\int_{L/2}^L x\,dx = \frac{1}{L}\left(\frac{x^2}{2}\Big|_{L/2}^L\right) = \boxed{3L/8 = A_0}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{L/2}^L x\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2} - \frac{L}{n\pi}\int_{L/2}^L \sin\left(\frac{n\pi}{L}x\right)dx\right)$$

$$= \frac{2}{L}\left(\frac{-L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^L\right)$$

$$= \frac{2}{L}\left(\frac{-L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}\right)\right)$$

$$\boxed{A_n = \frac{1}{n\pi}\left(\frac{2}{n\pi}\cos(n\pi) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)}$$