Homework #12 Solutions
Math 182, Spring 2009
Instructor: Dr. Doreen DeLeon

$\boxed{A}$ Solve Laplace's equation $\Delta u=0$, $0<x<L$, $0<y<H$ subject to

1. $u_x(0,y)=0,\ u_x(L,y)=0,\ u(x,0)=0,\ u(x,H)=f(x)$

$$\begin{cases} \Delta u=0 \\ u_x(0,y)=u_x(L,y)=0 \\ u(x,0)=0 \\ u(x,H)=f(x) \end{cases}$$

Let $u(x,y)=\phi(x)G(y)$

Then $\dfrac{d^2\phi(x)}{dx^2}G(y)+\phi(x)\dfrac{d^2G(y)}{dy^2}=0$

$$\Rightarrow \left(\frac{d^2\phi(x)}{dx^2}G(y)=-\phi(x)\frac{d^2G(y)}{dy^2}\right)\frac{1}{\phi(x)G(y)}$$

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2}=-\frac{1}{G(y)}\frac{d^2G(y)}{dy^2}=-\lambda$$

So $\phi''(x)=-\lambda\phi(x)$ and $G''(y)=\lambda G(y)$

Solve for $\phi(x)$: $\phi''+\lambda\phi=0$

$\phi'(0)=0,\ \phi'(L)=0$ (since $0=u_x(0,y)=\phi'(0)G(y)\Rightarrow\phi'(0)=0$; similarly for $\phi'(L)=0$).

From our work with the heat equation, we know

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ \phi_n(x)=\cos\left(\frac{n\pi}{L}x\right),\ n=1,2,3,\dots$$

$$\lambda_0=0,\qquad \phi_0(x)=1$$

Solve for $G(y)$: $G''-\lambda G=0$, $G(0)=0$ (since $0=u(x,0)=\phi(x)G(0)$)

$\lambda=0$: $G=c_1+c_2y$

$G(0)=0\Rightarrow c_1=0 \Rightarrow G_0(y)=y$ (let $c_2=1$)

$\lambda>0$: $G_n=c_1\cosh(\sqrt{\lambda}\,y)+c_2\sinh(\sqrt{\lambda}\,y)$

$G(0)=0\Rightarrow c_1=0 \Rightarrow G_n=\sinh(\sqrt{\lambda}\,y)=\sinh(\frac{n\pi}{L}y)$ (let $c_2=1$)

So, by superposition: $u(x,y)=A_0y+\sum_{n=1}^{\infty}A_n\sinh(\frac{n\pi}{L}y)\cos(\frac{n\pi}{L}x)$

$u(x,H)=f(x)\Rightarrow\ f(x)=A_0H+\sum_{n=1}^{\infty}A_n\sinh(\frac{n\pi}{L}H)\cos(\frac{n\pi}{L}x)$

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A_0: Integrate from 0 to L:

$$\int_0^L f(x)dx = \int_0^L A_0 H dx + \int_0^L \left(\sum_{n=1}^{\infty} A_n \sinh\left(\tfrac{n\pi}{L}H\right)\cos\left(\tfrac{n\pi}{L}x\right)\right)dx$$

$$= A_0 HL + \sum_{n=1}^{\infty} A_n \sinh\left(\tfrac{n\pi}{L}H\right)\int_0^L \cos\left(\tfrac{n\pi}{L}x\right)dx$$

$$= A_0 HL$$

$$\Rightarrow A_0 = \frac{1}{HL}\int_0^L f(x)dx$$

A_n: Multiply by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate from 0 to L

$$\int_0^L f(x)\cos\left(\tfrac{m\pi}{L}x\right)dx = \int_0^L A_0 H\cos\left(\tfrac{m\pi}{L}x\right)dx + \int_0^L \left(\sum_{n=1}^{\infty} A_n \sinh\left(\tfrac{n\pi}{L}H\right)\cos\left(\tfrac{n\pi}{L}x\right)\cos\left(\tfrac{m\pi}{L}x\right)\right)$$

$$= \sum_{n=1}^{\infty} A_n \sinh\left(\tfrac{n\pi}{L}H\right)\int_0^L \cos\left(\tfrac{n\pi}{L}x\right)\cos\left(\tfrac{m\pi}{L}x\right)dx$$

$$= A_m \sinh\left(\tfrac{m\pi}{L}H\right)\cdot L/2$$

$$\Rightarrow A_m = 2/\left(L\sinh\left(\tfrac{m\pi}{L}H\right)\right)\cdot\int_0^L f(x)\cos\left(\tfrac{m\pi}{L}x\right)dx$$

So $\boxed{u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\tfrac{n\pi}{L}y\right)\cos\left(\tfrac{n\pi}{L}x\right)\text{ where } A_0 = \tfrac{1}{LH}\int_0^L f(x)dx \text{ and } A_n = 2/\left(L\sinh\left(\tfrac{n\pi}{L}H\right)\right)\int_0^L f(x)\cos\left(\tfrac{n\pi}{L}x\right)dx}$

2. $u(0,y)=0,\ u(L,y)=0,\ u(x,0)-u_y(x,0)=0,\ u(x,H)=f(x)$

$$\begin{cases} \Delta u = 0 \\ u(0,y)=u(L,y)=0 \\ u(x,0)-u_y(x,0)=0 \\ u(x,H)=f(x) \end{cases}$$

Let $u(x,y)=\phi(x)G(y)$

Then, in the same way as #1, we get

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = \frac{-1}{G(y)}\frac{d^2G(y)}{dy^2} = -\lambda$$

$\Rightarrow \phi''(x) = \lambda\phi(x)$ and $G''(y) = \lambda G(y)$

Solve for $\phi(x)$: $\phi''+\lambda\phi=0,\ \phi(0)=\phi(L)=0$

From our work with the heat equations

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2,\ \phi_n(x)=\sin\left(\frac{n\pi}{L}x\right),\ n=1,2,3,\ldots$$

Solve for $G(y)$: $G''-\lambda G=0,\ G(0)-G'(0)=0$ (From the boundary condition).

$$\lambda_n>0 \Rightarrow G_n = c_1\cosh(\sqrt{\lambda}y)+c_2\sinh(\sqrt{\lambda}y)$$

$$G'(y) = \sqrt{\lambda}c_1\sinh(\sqrt{\lambda}y)+\sqrt{\lambda}c_2\cosh(\sqrt{\lambda}y)$$

$$G(0)-G'(0)=0 \Rightarrow c_1-\sqrt{\lambda}c_2=0 \Rightarrow c_1=\sqrt{\lambda}c_2$$

$$=\left(\frac{n\pi}{L}\right)c_2$$

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Let $c_2 = 1 \Rightarrow c_1 = \frac{n\pi}{L}$

$\Rightarrow G_n(y) = \left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)$

By superposition, $u(x,y) = \sum_{n=1}^{\infty} B_n \phi_n(x) G_n(y)$

$= \sum_{n=1}^{\infty} B_n \left[\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right]\sin\left(\frac{n\pi}{L}x\right)$.

$u(x,H) = f(x)$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} B_n\, G_n(H)\sin\left(\frac{n\pi}{L}x\right)$

Multiply by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate from 0 to L:

$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \int_0^L\left(\sum_{n=1}^{\infty} B_n G_n(H)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\right)dx$

$= \sum_{n=1}^{\infty} B_n G_n(H)\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$

$= B_m G_n(H)\cdot L/2$

$B_m = \frac{2}{L G_n(H)}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$

So $\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\left(\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right)\right), \text{ where } B_n = \frac{2}{L\left(\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right)\right)}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$

3. $u_x(0,y) = 0,\ u_x(L,y) = 0,\ u(x,0) = \begin{cases} 0, & x > L/2 \\ 1, & x < L/2 \end{cases},\ u_y(x,H) = 0$

$$\begin{cases} \Delta u = 0 \\ u_x(0,y) = u_x(L,y) = 0 \\ u(x,0) = \begin{cases} 0, & x > L/2 \\ 1, & x < L/2 \end{cases} \\ u_y(x,H) = 0 \end{cases}$$

Let $u(x,y) = \phi(x)G(y)$

Then, as before, we obtain: $\frac{1}{\phi(x)}\frac{d^2\phi}{dx^2}(x) = \frac{-1}{G(y)}\frac{d^2G}{dy^2}(y) = -\lambda$

$\Rightarrow \phi''(x) = -\lambda\phi(x),\quad G''(y) = \lambda G(y)$

Solve for $\phi(x)$: $\phi'' + \lambda\phi = 0,\ \phi'(0) = \phi'(L) = 0$

$\Rightarrow \lambda_0 = 0,\ \phi_0(x) = 1$ (see #1)

$\lambda_n = \left(\frac{n\pi}{L}\right)^2,\ \phi_n(x) = \cos\left(\frac{n\pi}{L}x\right),\ n = 1, 2, \ldots$

Solve for $G(y)$: $G'' - \lambda G = 0,\ G'(H) = 0$

$\lambda_n > 0$: $G = c_1\cosh(\sqrt{\lambda}(y-H)) + c_2\sinh(\sqrt{\lambda}(y-H))$

$G' = \sqrt{\lambda}c_1\sinh(\sqrt{\lambda}(y-H)) + \sqrt{\lambda}c_2\cosh(\sqrt{\lambda}(y-H))$

$G'(H) = 0 \Rightarrow \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0$

So $G = c_1 \cosh(\sqrt{\lambda}(y-H))$

let $c_1 = 1 \Rightarrow G_n = \cosh(\frac{n\pi}{L}(y-H))$

$\lambda = 0 \Rightarrow G = c_1 + c_2 y$

$G' = c_2$

So $G'(H) = 0 \Rightarrow c_2 = 0$

$G = c_1$ let $c_1 = 1 \Rightarrow G_0 = 1$

So $u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x)\cosh(\frac{n\pi}{L}(y-H))$

$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x)\cosh(\frac{n\pi}{L}H) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$

Find A_0: Integrate from 0 to L

$\int_0^L u(x,0)\,dx = \int_0^L A_0\,dx + \int_0^L \left(\sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x)\cosh(\frac{n\pi}{L}H)\right)dx$

$\int_0^{L/2} 1\,dx = A_0 L + \sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{L}H)\int_0^L \cos(\frac{n\pi}{L}x)\,dx$

$L/2 = A_0 L \Rightarrow A_0 = 1/2$

Find A_n: Multiply by $\cos(\frac{m\pi}{L}x)$ and integrate from 0 to L

$\int_0^L u(x,0)\cos(\frac{m\pi}{L}x)\,dx = \int_0^L A_0 \cos(\frac{m\pi}{L}x)\,dx$

$+ \int_0^L \left(\sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{L}H)\cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)\right)d$

$\int_0^{L/2}\cos(\frac{m\pi}{L}x)\,dx = \sum_{n=1}^{\infty} A_n \int_0^L \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)\,dx$

$\frac{L}{m\pi}\sin(\frac{m\pi}{L}x)\Big|_0^{L/2} = A_m \cosh(\frac{m\pi}{L}H)\cdot L/2$

$\frac{L}{m\pi}\sin(\frac{m\pi}{2}) = A_m \cdot L/2 \cosh(\frac{m\pi}{L}H)$

$A_m = \frac{2}{m\pi\cosh(\frac{m\pi}{L}H)}\sin(\frac{m\pi}{2})$

So $\boxed{u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n \cosh(\frac{n\pi}{L}(y-H))\cos(\frac{n\pi}{L}x) \text{ where } A_n = \frac{2}{n\pi\cosh(\frac{n\pi}{L}H)}\cdot\sin(\frac{n\pi}{2})}$

[B] 7.1 Solve Laplace's equation $\Delta u = 0$ in the domain $x^2+y^2>4$, subject to the boundary condition $u(x,y)=y$ on $x^2+y^2=4$ and the decay condition $\lim_{|x|+|y|\to\infty} u(x,y)=0$.

Transform to polar coordinates (r,θ) where $0\le r<2$ and $-\pi\le\theta\le\pi$

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 & (a)\\ u(2,\theta)=2\sin\theta & (b)\\ u(r,-\pi)=u(r,\pi) & (c)\\ \partial u/\partial\theta(r,-\pi)=\partial u/\partial\theta(r,\pi) & (d)\\ \lim_{r\to\infty} u(r,\theta)=0 \end{cases}$$

We seek solutions of the form $u(r,\theta)=\Phi(\theta)G(r)$

$$\Rightarrow \quad \Delta u = \frac{\Phi}{r}\frac{d}{dr}(rG'(r))+\frac{G}{r^2}\Phi''(\theta)=0$$

Divide by $G\Phi/r^2$ to obtain:

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=\frac{-1}{\Phi}\frac{d^2\Phi(\theta)}{d\theta^2}=\lambda$$

(c) + (d) give $\Phi(-\pi)=\Phi(\pi)$, $\Phi'(-\pi)=\Phi'(\pi)$

So, we must solve:

$$\begin{cases}\Phi''=-\lambda\Phi, \ -\pi<\theta<\pi\\ \Phi(-\pi)=\Phi(\pi)\\ \Phi'(-\pi)=\Phi'(\pi)\end{cases} \text{ and } \begin{cases} r(rG')'=\lambda G, \ 0\le r<2\\ \lim_{r\to\infty} G(r)=0\end{cases}$$

1) From class, we know: $\lambda_0=0\to\Phi_0=1$

$\lambda_n=n^2$, $\Phi_n=\cos(n\theta)$, $\Psi_n=\sin(n\theta)$

2) $r(rG')'=\lambda G$

$r^2G''+rG'-\lambda G=0$

Try $G=r^p$

$\Rightarrow r^p(p(p-1)+p-\lambda)=0$

$p^2-\lambda=0$

$\lambda\ne0$: $p^2-n^2=0\Rightarrow p=\pm n$

So $G(r)=c_1r^n+c_2r^{-n}$

But, need $\lim_{r\to\infty}G(r)=0\Rightarrow c_1=0$ So $G=c_2r^{-n}\Rightarrow G_n=r^{-n}$

$\lambda = 0$: $r(rG')' = 0$

$\frac{d}{dr}(rG') = 0$

$rG' = c_1$

$G' = c_1/r \Rightarrow G = c_1 \ln r + c_2$

Need $\lim_{r \to \infty} G(r) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$

So $G = c_2 \Rightarrow G_0 = 0$

Use superposition:

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

$$u(2,\theta) = 2\sin\theta = \sum_{n=1}^{\infty} A_n 2^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n 2^{-n} \sin(n\theta)$$

Find A_n, B_n:

$$A_n = \frac{1}{\pi \cdot 2^{-n}} \int_{-\pi}^{\pi} 2\sin(\theta)\cos(n\theta)\,d\theta = 0$$

$$B_n = \frac{1}{\pi \cdot 2^{-n}} \int_{-\pi}^{\pi} 2\sin(\theta)\sin(n\theta)\,d\theta = \begin{cases} 0 & n \neq 1 \\ 2\left(\frac{2}{\pi} \cdot \pi\right) = 4, & n = 1 \end{cases}$$

So $\boxed{u(r,\theta) = 4 r^{-1} \sin(\theta)}$

or, in cartesian coordinates $\left(\frac{4}{r} \sin(\theta) = \frac{4 r \sin\theta}{r^2} \right)$

$$u(x,y) = \frac{4y}{x^2 + y^2}$$

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