Homework #1 Solutions
Math 182, Spring 2009
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1.1b) Show the following equation has a solution of the form $u(x,y)=f(ax+by)$ for proper choice of constants a and b. Find a, b.

$$3u_x - 7u_y = 0$$

Let $u(x,y) = f(ax+by)$

$$u_x = \frac{d}{dx} f(ax+by) = f'(ax+by) \cdot \frac{d(ax+by)}{dx} = af'$$

$$u_y = \frac{d}{dy} f(ax+by) = f'(ax+by) \cdot \frac{d}{dy}(ax+by) = bf'$$

So we have $3af' - 7bf' = 0$

$$(3a-7b)f' = 0$$

For this to be true for arbitrary differentiable f, need

$$\boxed{3a-7b=0}$$

1.2 Show that each of the following equations has a solution of the form $u(x,y)=e^{\alpha x+\beta y}$. Find the constants α and β for each.

* b) $u_{xx} + u_{yy} = 5e^{x-2y}$

Let $u(x,y) = e^{\alpha x+\beta y}$

$$u_x = \frac{d}{dx}(e^{\alpha x+\beta y}) = \alpha e^{\alpha x+\beta y}$$

$$u_{xx} = \frac{d^2}{dx^2}(e^{\alpha x+\beta y}) = \alpha^2 e^{\alpha x+\beta y}$$

Similarly, $u_{yy} = \beta^2 e^{\alpha x+\beta y}$

Plug into the PDE to obtain

$$\alpha^2 e^{\alpha x+\beta y} + \beta^2 e^{\alpha x+\beta y} = 5e^{x-2y}$$

$$(\alpha^2+\beta^2)e^{\alpha x+\beta y} = 5e^{x-2y}$$

For $e^{\alpha x+\beta y}$ to be a solution, clearly we need

$$\boxed{\alpha=1 \text{ and } \beta=-2}$$

Then $\alpha^2+\beta^2=5$, so $\boxed{u(x,y)=e^{x-2y}}$ is the solution

c) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$

Let $u(x,y) = e^{\alpha x+\beta y}$. Then $u_{xxxx} = \alpha^4 e^{\alpha x+\beta y}$,

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1-2c)(cont.) $u_{yyyy} = \beta^4 e^{\alpha x+\beta y}$, and $u_{xxyy} = \alpha^2\beta^2 e^{\alpha x+\beta y}$
(You should verify these.)

Plugging into the PDE:

$$\alpha^4 e^{\alpha x+\beta y} + \beta^4 e^{\alpha x+\beta y} + 2\alpha^2\beta^2 e^{\alpha x+\beta y} = 0$$

$$(\alpha^4+\beta^4+2\alpha^2\beta^2)e^{\alpha x+\beta y} = 0$$

$$\Rightarrow \alpha^4+\beta^4+2\alpha^2\beta^2 = 0$$

$$\Rightarrow (\alpha^2+\beta^2)^2 = 0$$

$$\Rightarrow \boxed{\alpha^2+\beta^2=0} \Rightarrow \alpha=\beta=0$$

*1.4a) Let $u(x,y) = h(\sqrt{x^2+y^2})$ be a solution of the minimal surface equation. Show that $h(r)$ satisfies the ODE

$$rh'' + h'(1+(h')^2) = 0$$

The minimal surface equation is

$$(1+u_y^2)\,u_{xx} - 2u_xu_y\,u_{xy} + (1+u_x^2)u_{yy} = 0$$

Since $u(x,y) = h(\sqrt{x^2+y^2})$

$$u_x = h'\cdot d/dx(\sqrt{x^2+y^2}) = \frac{x}{\sqrt{x^2+y^2}}\,h'$$

$$u_y = h'\cdot d/dy(\sqrt{x^2+y^2}) = y/(\sqrt{x^2+y^2})\cdot h'$$

$$u_{xx} = h''\cdot\frac{x^2}{x^2+y^2} - \frac{h'\cdot x^2}{(x^2+y^2)^{3/2}} + \frac{h'}{\sqrt{x^2+y^2}}$$

$$u_{xy} = h''\cdot\frac{xy}{x^2+y^2} - \frac{h'\cdot xy}{(x^2+y^2)^{3/2}}$$

$$u_{yy} = h''\frac{y^2}{x^2+y^2} - \frac{h'\cdot y^2}{(x^2+y^2)^{3/2}} + \frac{h'}{\sqrt{x^2+y^2}}$$

Plugging into the PDE gives

$$0 = \left(1 + h'y^2/\sqrt{x^2+y^2}\right)\cdot\left[h''\frac{x^2}{x^2+y^2} - \frac{h'x^2}{(x^2+y^2)^{3/2}} + \frac{h'}{\sqrt{x^2+y^2}}\right]$$

$$-2\cdot\frac{x}{\sqrt{x^2+y^2}}\cdot h'\cdot\frac{y}{\sqrt{x^2+y^2}}\cdot h'\cdot\left[h''\frac{xy}{x^2+y^2} - \frac{h'xy}{(x^2+y^2)^{3/2}}\right]$$

$$+\left(1+\frac{h'x^2}{\sqrt{x^2+y^2}}\right)\left[h''\frac{y^2}{x^2+y^2} - \frac{h'y^2}{(x^2+y^2)^{3/2}} + \frac{h'}{\sqrt{x^2+y^2}}\right]$$

1.4a) (cont.) $$= \frac{h'}{\sqrt{x^2+y^2}} + h'' + \frac{(h')^3}{\sqrt{x^2+y^2}}$$

Multiply both sides by $\sqrt{x^2+y^2}$

$$h''\sqrt{x^2+y^2} + h' + (h')^3 = 0$$

Let $r = \sqrt{x^2+y^2} \Rightarrow rh'' + h'(1+(h')^2) = 0$ ✓

* 1.5 Let $p: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that

$$u_t = p(u)u_x, \quad t>0$$

has a solution satisfying $u = f(x+p(u)t)$, where f is differentiable. Then, find solutions for the following

a) $u_t = Ku_x$

b) $u_t = uu_x$

c) $u_t = u\sin(u)u_x$

Differentiate $u = f(x+p(u)t)$ by t:

$$u_t = f'(x+p(u)t) \circ \frac{d}{dt}[x+p(u)t]$$

$$= f' \circ [p(u) + t \cdot p'(u) \cdot u_t]$$

$$\Rightarrow u_t = f'p(u) + tf'p'(u)u_t$$

$$\Rightarrow u_t(1-tf'p'(u)) = f'p(u)$$

$1-tf'p'(u) \neq 0$ on any t-interval, because then $f'p(u)=0$. But this is not possible because, if either p or f' vanishes in this interval, then $tf'p'=0$ there.

Therefore, $u_t = \dfrac{f'p(u)}{1-tf'p'(u)}$

Similarly, $u_x = \dfrac{f'}{1-tf'p'(u)}$

Thus, $u_t = p(u)u_x$.

a) $p(u)=K$, so $u(x,t) = f(x+Kt)$, where f is any differentiable function

b) $p(u)=u$, so $u(x,t) = f(x+ut)$, where f is any differentiable function

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1.5 c) $p(u) = u\sin(u) \Rightarrow u(x,t) = f(x + u\sin(u)t)$, where f is any differentiable function

1.7 a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write the equation in the coordinates $s = x$, $t = x - y$.

b) Find the general solution of the equation

c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it in the coordinates $s = x + y$, $t = 2x$.

a) Let $v(s,t) = u(x,y)$

Then $u_x = \frac{\partial v}{\partial s} \cdot \frac{ds}{dx} + \frac{\partial v}{\partial y} \cdot \frac{dt}{dx} = v_s + v_y$

$u_y = \frac{\partial v}{\partial s} \cdot \frac{ds}{dy} + \frac{\partial v}{\partial s} \cdot \frac{dt}{dy} = -v_t$

Similarly, we can determine that

$u_{xx} = v_{ss} + v_{tt} + 2v_{st}$, $u_{xy} = -v_{tt} - v_{st}$, $u_{yy} = v_{tt}$

Plug into the PDE:

$v_{ss} + v_{tt} + 2v_{st} + 2(-v_{tt} - v_{st}) + v_{tt} = 0$

$\boxed{v_{ss} = 0}$

b) $v_{ss} = 0 \rightarrow \int v_{ss}\,ds = \int 0\,ds$

$\int v_{st} = \int f(t)\,ds$

$v = sf(t) + g(t)$, where f and g are arbitrary differentiable functions

$\Rightarrow \boxed{u(x,y) = f(x-y) + xg(x-y)}$

c) Let $v(s,t) = u(x,y)$

$\Rightarrow u_x = v_s + 2v_t$, $u_y = v_s$

$u_{xx} = v_{ss} + 4v_{tt} + 4v_{st}$, $u_{xy} = v_{ss} + 2v_{st}$, $u_{yy} = v_{ss}$

Plug into the PDE:

$v_{ss} + 4v_{tt} + 4v_{st} - 2(v_{ss} + 2v_{st}) + 5v_{ss} = 0$

$4(v_{ss} + v_{tt}) = 0$

$\Rightarrow \boxed{v_{ss} + v_{tt} = 0}$

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