Homework #3(a) Solutions
Math 182, Spring 2009
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1. Solve $u_y + uu_x = 0$, $u(x,0) = \begin{cases} 1, & x \le 0 \\ 1-\frac{1}{2}x, & 0<x<2 \\ 0, & x \ge 2 \end{cases}$

- Since $u(x,0)$ is not monotone increasing, the solution will develop a singularity at some positive time,
$y_c = -1/h'(s) = \frac{-1}{-\frac{1}{2}} = 2$ $\quad (h(x) = u(x,0))$.

For all $y < 2$:

$dx/dt = u \Rightarrow x = s + h(s)y$
$dy/dt = 1 \Rightarrow y = t$
$du/dt = 0 \Rightarrow u(t,s) = h(s)$

$u = h(s) = 1$ if $s \le 0$: $x = s + h(s)y = s + y$, $s \le 0 \Rightarrow x \le y$
$u = h(s) = 0$ if $s \ge 2$: $x = s$; $s \ge 2 \Rightarrow x \ge 2$

So, since $u = h(s) = h(x - uy)$,

if $x \le y$, $u = 1$
if $x \ge 2$, $u = 0$ $\Big\}$ as above
if $y < x < 2$: $u = h(s) = 1 - \frac{1}{2}s \Rightarrow u = 1 - \frac{1}{2}(x - uy)$
$\Rightarrow u = (1 - \frac{1}{2}x)/(1 - \frac{1}{2}y) = \frac{(\frac{1}{2}x - 1)}{(\frac{1}{2}x - 2)}$
$= (x-2)/(y-2)$

So $u(x,y) = \begin{cases} 1 & x < y \\ (x-2)/(y-2) & y < x < 2 \\ 0 & x \ge 2 \end{cases}$

After $y_c = 2$, need to determine a weak solution.
Seek a solution with one discontinuity, γ.
Our previous work tells us that γ moves with speed
$\gamma_y = \frac{1}{2}(u^- + u^+) = \frac{1}{2}(0+1) = \frac{1}{2}$

The weak solution is given by
$u(x,y) = \begin{cases} 1 & x < \gamma(y) \\ 0 & x > \gamma(y) \end{cases}$

$\gamma_y = \frac{1}{2}$ and the shock starts at $y_c = 2$, $x = 2 = \gamma(y_c)$

So $\int_2^y \gamma_y \, dy = \int_2^y \frac{1}{2} \, dy \Rightarrow \gamma(y) - 2 = \frac{1}{2}(y-2)$
$\Rightarrow \gamma(y) = 2 + \frac{1}{2}(y-2)$

So $u(x,y) = \begin{cases} 1, & x < 2 + \frac{1}{2}(y-2) \\ 0, & x > 2 + \frac{1}{2}(y-2) \end{cases}$ for $y > 2$

2. Solve $u_y + u u_x = 0$, $u(x,0) = \begin{cases} 3, & x < 0 \\ 4, & x > 0 \end{cases}$ $(= h(x))$

Here, $h' \geq 0 \Rightarrow$ characteristics do not intersect.
Instead, they diverge $\Rightarrow$ have an expansion wave.
Characteristic equations give:

$$x = s + h(s)y, \quad u = h(x - uy)$$

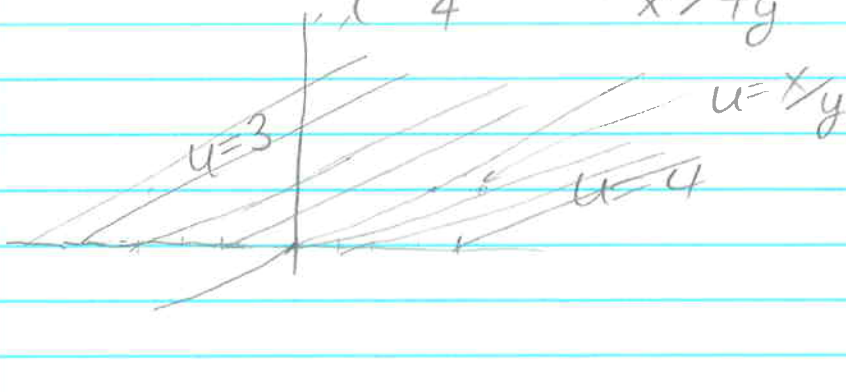
$h(s) = 3$ if $s < 0$: $x = s + 3y$; $s < 0 \Rightarrow x < 3y$
$h(s) = 4$ if $s > 0$: $x = s + 4y$; $s > 0 \Rightarrow x > 4y$

Need to obtain a solution for $x < 3y$, $3y \leq x \leq 4y$, $x > 4y$

$u = h(x - uy)$: $u = 3$ if $x < 3y$, $u = 4$ if $x > 4y$ from above

Assume all values of u between 3 and 4 are present initially at $x = 0 \Rightarrow$ there is a straight-line characteristic along which u takes each value between 3 and 4. The characteristic starts at $x = 0, y = 0$ (so $s = 0$) $\Rightarrow x - uy = 0 \Rightarrow x = uy$
$\Rightarrow u = x/y$

So $u(x,y) = \begin{cases} 3 & x < 3y \\ x/y & 3y \leq x \leq 4y \\ 4 & x > 4y \end{cases}$



3. Solve $u_y + 4u_x = 0$, $u(x,0) = \begin{cases} 4, & x<1 \\ 3, & x>1 \end{cases}$

Since $u(x,0)$ is decreasing, a shock will develop when the characteristics intersect

The characteristics satisfy $x = s + h(s)y$

$h(s) = 4$ if $s<1$: $x = s+4y$; $s<1 \Rightarrow x < 1+4y$
$h(s) = 3$ if $s>1$: $x = s+3y$; $s>1 \Rightarrow x > 1+3y$ $\Big\}$ intersect at $y=0$

So the discontinuity is immediate

$\Rightarrow u(x,y) = \begin{cases} 4, & x<\gamma(y) \\ 3, & x>\gamma(y) \end{cases}$

$\gamma_y = \frac{1}{2}(u^- + u^+) = \frac{1}{2}(4+3) = 7/2$

Since the shock starts at $x=1, y=0$: $\gamma(0) = 1$

$\Rightarrow \gamma(y) - 1 = 7/2\, y \Rightarrow \gamma(y) = 7/2\, y + 1$

So $\boxed{u(x,y) = \begin{cases} 4, & x < 7/2\, y + 1 \\ 3, & x > 7/2\, y + 1 \end{cases}}$