Homework #3b) Solutions
Math 182, Spring 2009

1. Solve $u_x^2 + u_y^2 = 1$, $u(x, -\frac{1}{2}x) = 1$
   - Characteristic equations

$$\frac{d^2x}{dt^2} = \frac{1}{2}(1)_x = 0, \quad \frac{d^2y}{dt^2} = \frac{1}{2}(1)_y = 0$$

$\Rightarrow$ Characteristics are straight lines emanating from the initial line $y = -\frac{1}{2}x$

u is constant along $y = -\frac{1}{2}x \Rightarrow \nabla u$ is orthogonal to it

So $(u_x, y_y) \cdot (x, -\frac{1}{2}x) = 0$

$\Rightarrow (u_x, u_y) = c(\frac{1}{2}, 1)$

Since $u_x^2 + u_y^2 = 1$, we have

$(\frac{1}{2}c)^2 + c^2 = 1 \Rightarrow (\frac{5}{4})c^2 = 1 \Rightarrow c = \pm 2/\sqrt{5}$

Let $c = 2/\sqrt{5} \Rightarrow (u_x, u_y) = (\frac{1}{\sqrt{5}}, 2/\sqrt{5})$

Then $\frac{dx}{dt}(0) = \frac{1}{\sqrt{5}}, \quad \frac{dy}{dt}(0) = \frac{2}{\sqrt{5}}$ (since $x_t(0) = u_x$, $y_t(0) = u_y$)

Initial conditions: $x(0,s) = s$, $y(0,s) = -\frac{1}{2}s$, $u(0,s) = 1$

$x_{tt} = 0 \Rightarrow x_t = \frac{1}{\sqrt{5}} \Rightarrow x = \frac{1}{\sqrt{5}}t + f_1(s)$; $x(0,s) = s \rightarrow x = \frac{1}{\sqrt{5}}t + s$ (a)

$y_{tt} = 0 \rightarrow y_t = 2/\sqrt{5} \Rightarrow y = \frac{2}{\sqrt{5}}t + f_2(s)$; $y(0,s) = -\frac{1}{2}s \Rightarrow y = \frac{2}{\sqrt{5}}t - \frac{1}{2}s$ (b)

$u_t = 1 \rightarrow u = t + f_3(s)$; $u(0,s) = 1 \rightarrow u(t,s) = t + 1$

Solve for t: (a) + 2(b) $\Rightarrow x + 2y = \sqrt{5}t \Rightarrow t = \frac{x+2y}{\sqrt{5}}$

So, $\boxed{u(x,y) = 1 + \frac{x+2y}{\sqrt{5}}}$

2. Solve $u_x^2 + u_y^2 = 1$, $u(1,y) = \sqrt{1+y^2}$

Initial conditions:

$x(0,s) = 1$, $y(0,s) = s$, $u(0,s) = \sqrt{1+s^2}$

Characteristic equations

$x_{tt} = \frac{1}{2}(1)_x = 0$, $y_{tt} = \frac{1}{2}(1)_y = 0$, $u_t = 1$

$u_t = 1 \Rightarrow u = t + u_0(s) \rightarrow u = t + \sqrt{1+s^2}$

Need $x_t(0,s)$, $y_t(0,s)$ but $x_t(0,s) = u_x(0,s)$, $y_t(0,s) = u_y(0,s)$

$u_y = u_s \Rightarrow u_y = \frac{s}{\sqrt{1+s^2}}$

Using the PDE, $u_x^2 = 1 - u_y^2 \Rightarrow u_x = \frac{1}{\sqrt{1+s^2}}$

So $x_t(0,s) = u_x = 1/\sqrt{1+s^2}$, $y_t(0,s) = u_y = s/\sqrt{1+s^2}$

$x_{tt} = 0 \Rightarrow x_t = f_1(s) \rightarrow x = f_1(s)t + f_2(s)$

$x(0,s) = 1 \rightarrow x = f_1(s)t + 1$

$x_t(0,s) = 1/\sqrt{1+s^2} \rightarrow x = t/\sqrt{1+s^2} + 1$

$y_{tt} = 0 \Rightarrow y = f_3(s)t + f_4(s)$ and $y(0,s) = s \rightarrow y = f_3(s)t + s$

$y_t(0,s) = s/\sqrt{1+s^2} \rightarrow y = st/\sqrt{1+s^2} + s$

Need to solve for t and s in terms of x and y:

$x^2 + y^2 = (t/\sqrt{1+s^2} + 1)^2 + (st/\sqrt{1+s^2} + s)^2$

$= (t + \sqrt{1+s^2})^2 \Rightarrow t + \sqrt{1+s^2} = \pm\sqrt{x^2+y^2}$

So, since $u(t,s) = t + \sqrt{1+s^2}$, and $u(1,y) > 0$:

$\boxed{u(x,y) = \sqrt{x^2+y^2}}$

Homework #4 Solutions
Math 182, Spring 2009
Instructor: Dr. Doreen De Leon

1. p. 73: 3.2 (a) Show that the following equation is hyperbolic

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0.$$

(b) Find the canonical form of the equation

(c) Find the general solution $u(x,y)$

(d) Find a solution $u(x,y)$ that satisfies $u(-x,2x) = x$ and $u(x,0) = \sin(2x)$

(a) $b = 3,\ a = 1,\ c = -16 \rightarrow \delta(L) = 3^2 - (1)(-16) = 25 > 0 \Rightarrow$ hyperbolic

(b) $$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{3 \pm \sqrt{25}}{1} = 8, -2$$

$+$: $dy/dx = 8 \rightarrow y = 8x + c \Rightarrow y - 8x = c$

$-$: $dy/dx = -2 \rightarrow y = -2x + c \Rightarrow y + 2x = c$

So $\xi = y - 8x$, $\eta = y + 2x$

Then $u_x = -8u_\xi + 2u_\eta$ $\qquad$ $u_y = u_\xi + u_\eta$

$u_{xx} = -8(-8u_\xi + 2u_\eta)_\xi + 2(-8u_\xi + 2u_\eta)_\eta$

$= 64u_{\xi\xi} - 32u_{\xi\eta} - 12u_{\eta\eta}$

$u_{yy} = (u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta$

$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = (-8u_\xi + 2u_\eta)_\xi + (-8u_\xi + 2u_\eta)_\eta$

$= -8u_{\xi\xi} - 6u_{\xi\eta} + 2u_{\eta\eta}$

Plug in:

$\cancel{64u_{\xi\xi}} - 32u_{\xi\eta} - 12u_{\eta\eta} + 6(\cancel{-8u_{\xi\xi}} - 6u_{\xi\eta} + 2u_{\eta\eta})$

$-16(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 0$

$-100u_{\xi\eta} = 0 \rightarrow u_{\xi\eta} = 0$

(c) $u_{\xi\eta} = 0 \Rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$

$\Rightarrow u(x,y) = F(y - 8x) + G(y + 2x)$

(d) $u(-x,2x) = x \Rightarrow x = F(2x - 8(-x)) + G(2x - 2x)$

$x = F(10x) + G(0)$

$u(x,0) = \sin(2x) \Rightarrow \sin(2x) = F(-8x) + G(2x)$

Suppose $G(0) = 0 \Rightarrow F(10x) = x \Rightarrow F(z) = \frac{1}{10}z$

Then $F(-8x) = \frac{-8}{10}x = \frac{-4}{5}x \Rightarrow \frac{-4}{5}x + G(2x) = \sin(2x)$

$G(2x) = \sin(2x) + \frac{4}{5}x \Rightarrow G(z) = \sin(z) + \frac{2}{5}z$ (Note: $G(0) = 0$ ✓)

So $u(x,y) = F(y-8x) + G(y+2x)$

$= \frac{1}{10}(y-8x) + \sin(y+2x) + \frac{2}{5}(y+2x)$

$$\boxed{u(x,y) = \frac{1}{2}y + \sin(y+2x)}$$

2. p73, 3-4 Consider the equation

$$y^5 u_{xx} - y u_{yy} + 2u_y = 0, \quad y > 0$$

(a) Find the canonical form of the equation

(b) Find the general solution $u(x,y)$ of the equation.

(c) Find the solution $u(x,y)$ which satisfies $u(0,y) = 8y^3$, $u_x(0,y) = 6$, $y > 0$.

(a) $b = 0,\ a = y^5,\ c = -y \Rightarrow \delta(L) = y^6 > 0$

$\frac{dy}{dx} = \frac{\pm\sqrt{y^6}}{y^5} = \pm y^{-2} \Rightarrow \int y^2\, dy/dx = \int \pm dx$

$\frac{1}{3}y^3 = \pm x + C \Rightarrow C = \frac{1}{3}y^3 \mp x$

So let $\xi = \frac{1}{3}y^3 - x$, $\eta = \frac{1}{3}y^3 + x$

$u_x = -u_\xi + u_\eta$

$u_{xx} = -(-u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta$

$= u_{\xi\xi} + u_{\eta\eta}$

$u_y = y^2 u_\xi + y^2 u_\eta$

$u_{yy} = 2y u_\xi + 2y u_\eta + y^2(y^2 u_\xi + y^2 u_\eta)_\xi + y^2(y^2 u_\xi + y^2 u_\eta)_\eta$

$= 2y u_\xi + 2y u_\eta + y^4 u_{\xi\xi} + 2y^4 u_{\xi\eta} + y^4 u_{\eta\eta}$

Plug in: $y^5(\cancel{u_{\xi\xi}} + \cancel{u_{\eta\eta}}) - y(2y u_\xi + 2y u_\eta + \cancel{y^4 u_{\xi\xi}} + 2y^4 u_{\xi\eta} + \cancel{y^4 u_{\eta\eta}}) + 2[y^2 u_\xi + y^2 u_\eta] = 0$

$-2y^5 u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$ (since $y > 0$)

(b) So $u(\xi,\eta) = F(\xi) + G(\eta)$

$\Rightarrow u(x,y) = F(\frac{1}{3}y^3 - x) + G(\frac{1}{3}y^3 + x)$

(c) $u(0,y) = 8y^3 \Rightarrow 8y^3 = F(\frac{1}{3}y^3) + G(\frac{1}{3}y^3)$ (a)

$u_x(0,y) = 6$: $u_x(x,y) = -F'(\frac{1}{3}y^3 - x) + G'(\frac{1}{3}y^3 + x)$

$\Rightarrow 6 = -F'(\frac{1}{3}y^3) + G'(\frac{1}{3}y^3)$

Integrate: $\int_0^y 6\, d(\frac{1}{3}s^3) = -F(\frac{1}{3}y^3) + F(0) + G(\frac{1}{3}y^3) - G(0)$

$2y^3 = -F(\frac{1}{3}y^3) + G(\frac{1}{3}y^3) + F(0) - G(0)$

Add (a)+(b) to get $10y^3 = 2G(\frac{1}{3}y^3) + F(0) - G(0)$

If we assume $F(0)=G(0)=0$, then

$$16y^3 = 2G(\tfrac{1}{3}y^3) \Rightarrow G(\tfrac{1}{3}y^3) = 5y^3$$
$$G(u) = 15u$$

Then $8y^3 = F(\tfrac{1}{3}y^3) + G(\tfrac{1}{3}y^3) \Rightarrow F(\tfrac{1}{3}y^3) = 8y^3 - 5y^3 = 3y^3$

$$F(u) = 9u$$

So $u(x,y) = F(\tfrac{1}{3}y^3 - x) + G(\tfrac{1}{3}y^3 + x)$

$$= 9(\tfrac{1}{3}y^3 - x) + 15(\tfrac{1}{3}y^3 + x)$$
$$= 8y^3 + 6x$$

$\boxed{u(x,y) = 8y^3 + 6x}$ (Verify this solves the PDE and satisfies the conditions)

3. Consider the PDE

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$

(a) Find the canonical form of the equation

(b) Find the solution $u(x,y)$ satisfying $u(0,y) = y^2$, $u_x(0,y) = 1-3y$

(a) $b=-2, a=1, c=4 \rightarrow \delta(L) = 4-4=0 \Rightarrow$ parabolic

$$\frac{dy}{dx} = \frac{-2}{1} = -2 \Rightarrow y = -2x + c \Rightarrow y + 2x = c$$

So, let $\eta = y + 2x$. Then let $\xi = x$

$\Rightarrow u_x = 2u_\eta + u_\xi \qquad u_y = u_\eta$

$u_{xx} = 2(2u_\eta + u_\xi)_\eta + (2u_\eta + u_\xi)_\xi \qquad u_{yy} = u_{\eta\eta}$

$= 4u_{\eta\eta} + 4u_{\xi\eta} + u_{\xi\xi}$

$u_{xy} = (2u_\eta + u_\xi)_\eta = 2u_{\eta\eta} + u_{\xi\eta}$

Plug in: $4u_{\eta\eta} + 4u_{\xi\eta} + u_{\xi\xi} - 4(2u_{\eta\eta} + u_{\xi\eta}) + 4u_{\eta\eta} = 0$

$$u_{\xi\xi} = 0$$

(b)

$$\Rightarrow u = \xi\phi(\eta) + \psi(\eta)$$

or $u(x,y) = x\phi(y+2x) + \psi(y+2x)$

$u(0,y) = y^2 \Rightarrow \psi(y) = y^2$

$u_x(0,y) = 1-3y$: $u_x(x,y) = \phi(y+2x) + 2x\phi'(y+2x) + 2\psi'(y+2x)$

$$u_x(0,y) = 1-3y = \phi(y) + 2\psi'(y)$$

$\Rightarrow 1-3y = \phi(y) + 2\cdot 2y \Rightarrow \phi(y) = 1-7y$

So $\boxed{u(x,y) = x\cdot(1-7(y+2x)) + (y+2x)^2}$

4. p. 74: 3.10. Consider the equation

$$u_{xx} - 2u_{xy} + 4e^y = 0$$

(a) Find the canonical form of the equation

(b) Find the solution $u(x,y)$ which satisfies $u(0,y) = f(y)$ and $u_x(0,y) = g(y)$.

(a) $b=-1,\ a=1,\ c=0 \rightarrow \delta(L) = 1-1\cdot 0 = 1 > 0 \Rightarrow$ hyperbolic

$$\frac{dy}{dx} = \frac{b \pm\sqrt{b^2-ac}}{a} = \frac{-1\pm 1}{1} = -2, 0 : \quad y = -2x + C,\ y = C$$

So $\xi = y + 2x$ and $\eta = y$

$u_x = 2u_\xi \qquad\qquad u_y = u_\eta + u_\xi$

$u_{xx} = 4u_{\xi\xi}$

$u_{xy} = (2u_\xi)_\eta + (2u_\xi)_\xi$

$\quad = 2u_{\xi\eta} + 2u_{\xi\xi}$

Plug in: $4u_{\xi\xi} - 2(2u_{\xi\eta} + 2u_{\xi\xi}) + 4e^y = 0$

$-4u_{\xi\eta} + 4e^y = 0$

$-4u_{\xi\eta} + 4e^\eta = 0$

(b) $u_{\xi\eta} = e^\eta$

$u_\xi = e^\eta + \phi(\xi)$

$u = \xi e^\eta + F(\xi) + G(\eta) \quad (F = \int\phi(\xi)d\xi)$

$u(x,y) = e^y(2x+y) + F(2x+y) + G(y)$

Use initial conditions:

$u(0,y) = f(y) \Rightarrow f(y) = ye^y + F(y) + G(y)$

$u_x(0,y) = g(y)$: $u_x(x,y) = 2e^y + 2F'(2x+y)$

$g(y) = u_x(0,y) = 2e^y + 2F'(y) \Rightarrow F'(y) = \frac{1}{2}g(y) - e^y$

$F(y) = \frac{1}{2}\int_0^y g(s)ds - \int_0^y e^s ds = \frac{1}{2}\int_0^y g(s)ds - e^y + 1$

$G(y) = f(y) - ye^y - F(y) = f(y) - ye^y + e^y - 1 - \frac{1}{2}\int_0^y g(s)ds$

$\Rightarrow \boxed{u(x,y) = e^y(2x+y) + \frac{1}{2}\int_0^{2x+y} g(s)ds - e^{2x+y} + 1 + f(y) - ye^y + e^y - 1 - \frac{1}{2}\int_0^y g(s)ds}$