Homework #5 Solutions
Math 182, Spring 2009

1- Consider the PDE
$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$
Find the region(s) in which the equation is hyperbolic, parabolic or elliptic

$b = -1,\ a = y,\ c = x \rightarrow \delta(L) = 1 - xy$

So the equation is hyperbolic in the region $xy < 1$, parabolic on the hyperbola $xy = 1$, and elliptic in the regions $xy > 1$.

2. Do the same for
$$xu_{xx} + 2u_{xy} + xu_{yy} - xu_x + x^2yu_y = \sin(y)$$

$b = 1,\ a = x;\ c = x \Rightarrow \delta(L) = 1 - x^2$
$$= (1+x)(1-x)$$
So the PDE is hyperbolic in the region $-1 < x < 1$, parabolic for $x = -1$ and $x = 1$, and elliptic in the regions $x < -1$ and $x > 1$.

3. p75: 3.12. Consider the equation
$$u_{xx} + yu_{yy} = 0.$$
Find the canonical forms of the equation for the domain where the equation is hyperbolic and for the domain where it is elliptic.

$\delta(L) = -y \Rightarrow$ hyperbolic for $y < 0$, elliptic for $y > 0$

$y < 0$ hyperbolic so $dy/dx = \pm\sqrt{-y} \Rightarrow -2(-y)^{\frac{1}{2}} = \pm x + c \Rightarrow -2(-y)^{\frac{1}{2}} \mp x = c$

Let $\xi = -2(-y)^{\frac{1}{2}} - x,\ \eta = -2(-y)^{\frac{1}{2}} + x$

$u_x = -u_\xi + u_\eta$

$u_{xx} = -(-u_\xi + u_\eta)_\xi + (-u_\xi + u_\eta)_\eta$

$= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$u_y = (-y)^{-\frac{1}{2}} u_\xi + (-y)^{-\frac{1}{2}} u_\eta$

$$
\begin{aligned}
u_{yy} &= \tfrac{1}{2}(-y)^{-3/2} u_\xi + \tfrac{1}{2}(-y)^{-3/2} u_\eta \\
&\quad + (-y)^{-\frac{1}{2}}\left[(-y)^{-\frac{1}{2}} u_\xi + (-y)^{-\frac{1}{2}} u_\eta\right]_\xi \\
&\quad + (-y)^{-\frac{1}{2}}\left[(-y)^{-\frac{1}{2}} u_\xi + (-y)^{-\frac{1}{2}} u_\eta\right]_\eta \\
&= \tfrac{1}{2}(-y)^{-3/2}\left[u_\xi + u_\eta\right] \\
&\quad - y^{-1}\left[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right]
\end{aligned}
$$

Plug in: $u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} + y\left(\tfrac{1}{2}(-y)^{-3/2}\left[u_\xi + u_\eta\right] - y^{-1}\left[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right]\right) = 0$

$-4u_{\xi\eta} - \tfrac{1}{2}(-y)^{-\frac{1}{2}}(u_\xi + u_\eta) = 0$

Since $(-y)^{-\frac{1}{2}} = -\left(\frac{\eta+\xi}{4}\right)^{-1}$, $-4u_{\xi\eta} + \frac{1}{2}\left(\frac{\eta+\xi}{4}\right)^{-1}(u_\xi + u_\eta) = 0$

$\Rightarrow \boxed{u_{\xi\eta} - \frac{1}{2(\eta+\xi)}\cdot(u_\xi + u_\eta) = 0}$

$\boxed{y>0}$ elliptic, so look at $dy/dx = \pm\sqrt{y} = \pm i y^{\frac{1}{2}}$

$\Rightarrow 2y^{\frac{1}{2}} = \pm ix + C \Rightarrow 2y^{\frac{1}{2}} \mp ix = C$

Therefore, the canonical variables are

$q = 2y^{\frac{1}{2}}, \quad r = x$

$u_x = u_r$

$u_{xx} = u_{rr}$

$u_y = y^{-\frac{1}{2}} u_q$

$u_{yy} = -\frac{1}{2} y^{-3/2} u_q + y^{-1} u_{qq}$

Plug in:

$u_{rr} + y\left(-\frac{1}{2} y^{-3/2} u_q + y^{-1} u_{qq}\right) = 0$

$u_{rr} + u_{qq} - \frac{1}{2} y^{-\frac{1}{2}} u_q = 0$

$y^{\frac{1}{2}} = \frac{1}{2} q \Rightarrow y^{-\frac{1}{2}} = 2/q \Rightarrow -\frac{1}{2} y^{-\frac{1}{2}} = -q^{-1}$

So the canonical form of the PDE is

$$\boxed{u_{rr} + u_{qq} - q^{-1} u_q = 0}$$