Homework #6 Solutions
Math 182, Spring 2009
Instructor: Dr. Doreen De Leon

[A] 1. Solve $u_{tt} - 25u_{xx} = 0, \; -\infty < x < \infty, \; t > 0$

$$u(x,0) = \underbrace{\sin(x)}_{f(x)}, \quad u_t(x,0) = \underbrace{\sin(3x)}_{g(x)}$$

Using d'Alembert's formula: $(c = \sqrt{25} = 5)$

$$u(x,t) = \frac{1}{2}\left(f(x-5t) + f(x+5t)\right) + \frac{1}{2(5)}\int_{x-5t}^{x+5t} g(s)\,ds$$

$$= \frac{1}{2}\left(\sin(x-5t) + \sin(x+5t)\right) + \frac{1}{10}\int_{x-5t}^{x+5t} \sin(3s)\,ds$$

$$= \frac{1}{2}\left(\sin(x-5t) + \sin(x+5t)\right) + \frac{1}{30}\left(-\cos(3s)\Big|_{x-5t}^{x+5t}\right)$$

$$\boxed{u(x,t) = \frac{1}{2}\left(\sin(x-5t) + \sin(x+5t)\right) + \frac{1}{30}\left(\cos(x-5t) - \cos(x+5t)\right)}$$

* 2. Sketch the solution pulse for various times

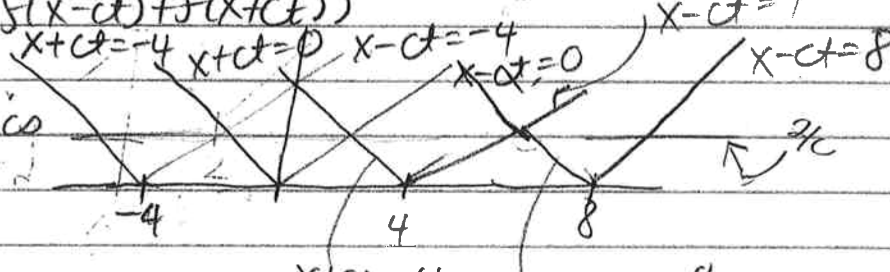
$$u_{tt} - c^2 u_{xx} = 0, \; -\infty < x < \infty, \; t > 0$$

$$u(x,0) = \begin{cases} 2, & -4 < x \le 0 \\ 3, & 4 \le x \le 8 \\ 0, & \text{otherwise} \end{cases}, \quad u_t(x,0) = 0 = g(x)$$

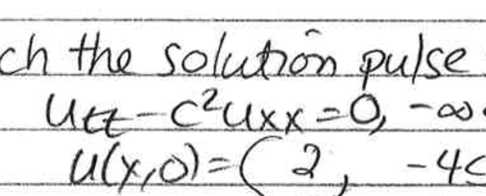
So $u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right)$$

characteristics



① $u(x,0)$



Endpoints: ① $-4 \le x - ct \le 0$

$-4 + ct \le x \le ct$

$-4 \le x + ct \le 0$

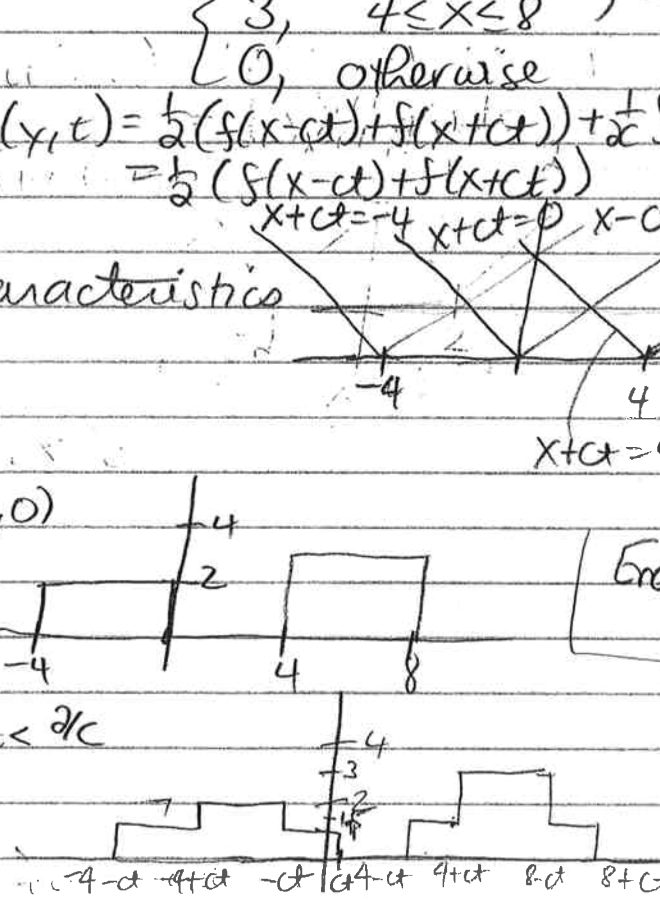
$-4 - ct \le x \le -ct$

② $4 \le x - ct \le 8$

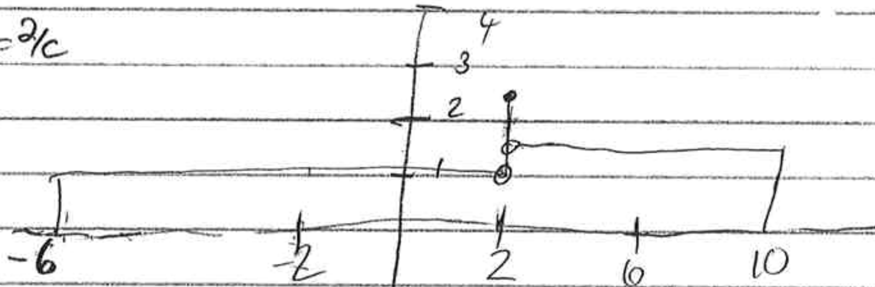
$4 + ct \le x \le 8 + ct$

and $4 - ct \le x \le 8 - ct$

② $0 < t < 2/c$

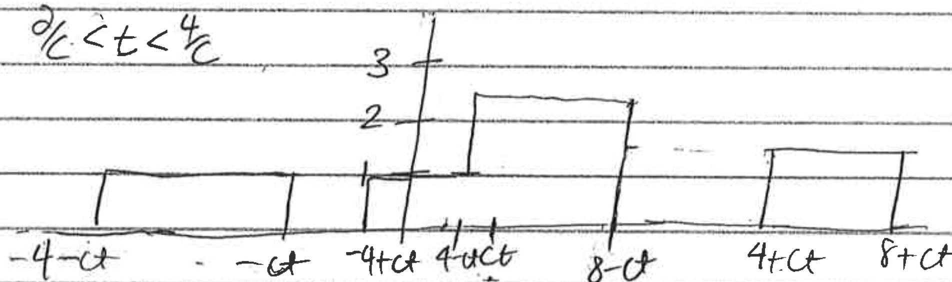


(3) $t = 2/c$

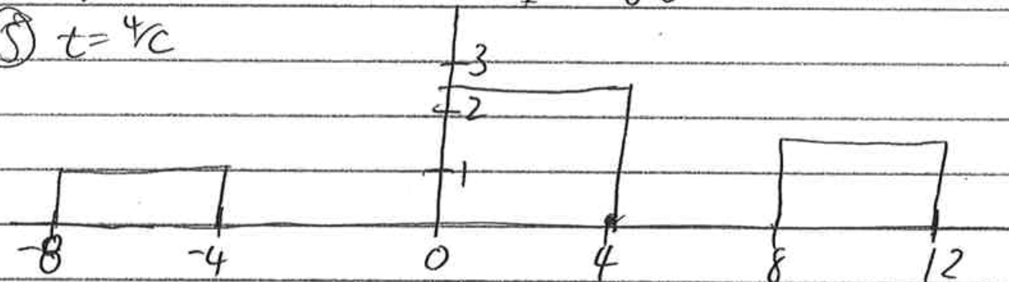


$-2 \le x \le 2 + -6 \le x \le -2$ $\quad$ $6 \le x \le 10 + 2 \le x \le 6$

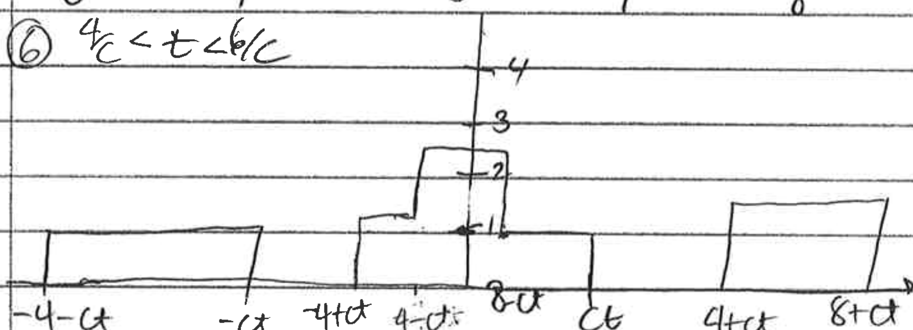
(4) $2/c < t < 4/c$



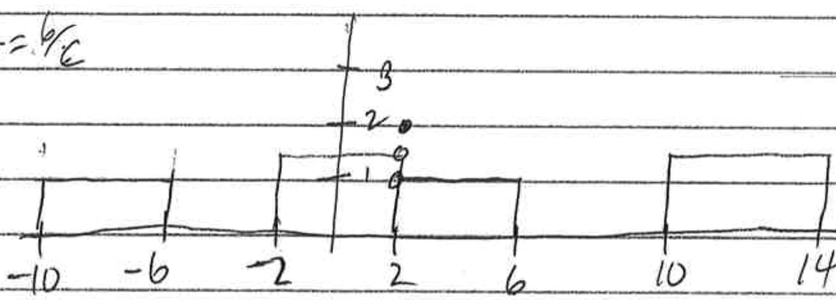
(5) $t = 4/c$



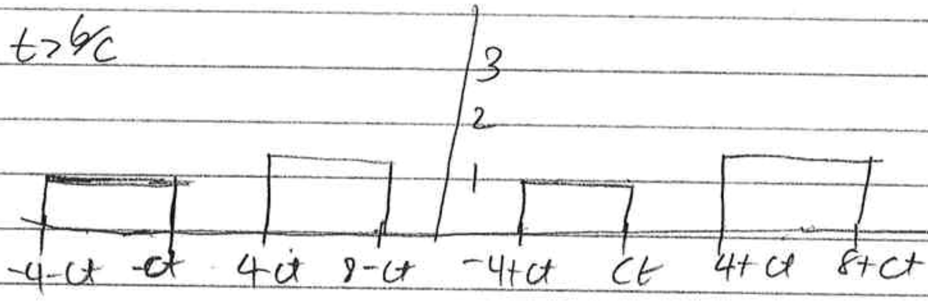
(6) $4/c < t < 6/c$



(7) $t = 6/c$



⑧ $t > 6/c$



$\boxed{B}$ *1. Determine analytic formulas for $u(x,t)$ if

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$\underbrace{u(x,0)}_{f(x)} = 0, \quad u_t(x,0) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h \end{cases} = g(x)$$

using characteristics.

From d'Alambert's formula:

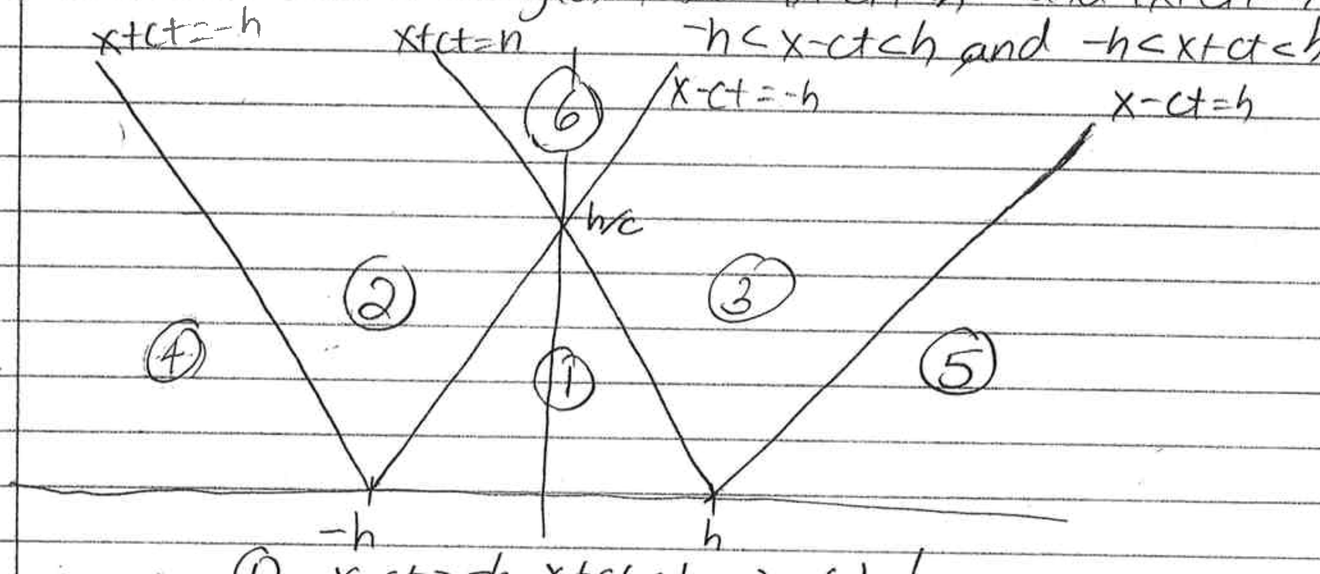
$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Since $f(x)=0$, $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

Where $g(s)=1$ for $|s|<h$

$\Rightarrow g(s)=1$ for $|x-ct|<h$ and $|x+ct|<h$

$-h < x-ct < h$ and $-h < x+ct < h$



① $x-ct > -h$, $x+ct < h \Rightarrow g(s)=1$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,ds = \frac{1}{2c}(x+ct-(x-ct)) = t$$

② $x+ct > -h$, $x+ct < h$, $x-ct < -h$

For $x-ct < -h$, $g(s)=0$ and $-h < x+ct < h$

$$\Rightarrow u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-h} g(s)\,ds + \int_{-h}^{x+ct} g(s)\,ds\right]$$

$$= \frac{1}{2c}\left[\int_{x-ct}^{-h} 0\,ds + \int_{-h}^{x+ct} 1\,ds\right]$$

$$= \frac{1}{2c}(h+x+ct)$$

③ $x-ct>-h$, $x-ct<h$, and $x+ct>h$

$\Rightarrow -h<x-ct<h$ (so $g(s)=1$), for $h<s<x+ct$, $g(s)=0$

$\Rightarrow u(x,t)=\frac{1}{2c}\left[\int_{x-ct}^{h} g(s)ds + \int_{h}^{x+ct} g(s)ds\right]$

$=\frac{1}{2c}\left[\int_{x-ct}^{h} 1ds + \int_{h}^{x+ct} 0ds\right]$

$=\frac{1}{2c}(h-x+ct)$

④ $x+ct<-h \Rightarrow x-ct<-h$

$\Rightarrow u(x,t)=0$

⑤ $x-ct>h \Rightarrow x+ct>h$

$\Rightarrow u(x,t)=0$

⑥ $x+ct>h$ and $x-ct<-h$

$\Rightarrow g(s)=0$ for $x-ct<s<-h$ and for $h<s<x+ct$

$\Rightarrow u(x,t)=\frac{1}{2c}\int_{-h}^{h} 1ds = h/c$

Summary:

$$u(x,t)=\begin{cases} t, & -h+ct<x<h-ct \\ \frac{1}{2c}(h+x+ct), & -h-ct<x<-h+ct \text{ and } x<h+ct \\ \frac{1}{2c}(h-x+ct), & -h+ct<x<h+ct \text{ and } x>h-ct \\ h/c & h-ct<x<-h+ct \\ 0 & \text{otherwise} \end{cases}$$

*2. Solve

$u_{tt}-c^2u_{xx}=0$, $-\infty<x<\infty$, $t>0$

$f(x)=u(x,0)=\begin{cases} |4-x|, & |x|\le 4 \\ 0, & |x|>4 \end{cases}$, $u_t(x,0)=0=g(x)$

By d'Alambert's formula:

$u(x,t)=\frac{1}{2}(f(x-ct)+f(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

$g(s)=0 \to u(x,t)=\frac{1}{2}(f(x-ct)+f(x+ct))$

$f(x-ct)=|4-(x-ct)|$ if $|x-ct|\le 4 \Rightarrow -4\le x-ct\le 4$

$f(x+ct)=|4-(x+ct)|$, if $|x+ct|\le 4 \Rightarrow -4\le x+ct\le 4$

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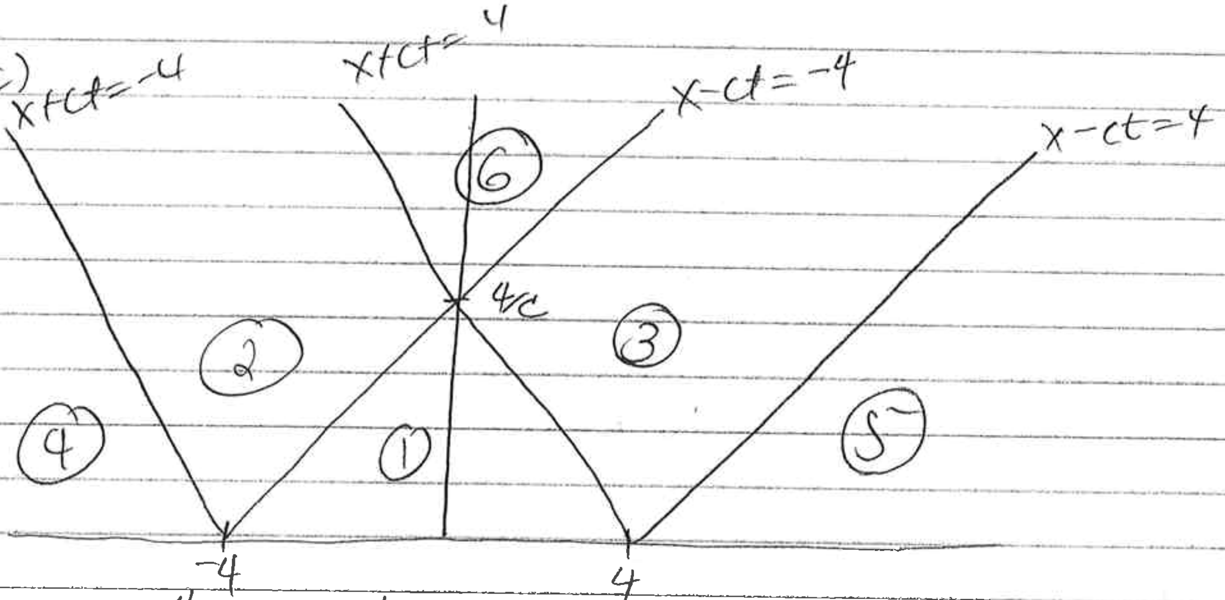
2. (cont.)



① $x-ct>-4$, $x+ct<4$ (so $x-ct<4$ and $-4<x+ct$)

$\Rightarrow u(x,t)=\frac{1}{2}(|4-x+ct|+|4-x-ct|)$

② $-4<x+ct<4$ and $x-ct<-4$

$\Rightarrow f(x-ct)=0,\ f(x+ct)=|4-x-ct|$

So $u(x,t)=\frac{1}{2}|4-x-ct|$

③ $-4<x-ct<4$ and $x+ct>4$

$\Rightarrow f(x-ct)=|4-x+ct|,\ f(x+ct)=0$

$\Rightarrow u(x,t)=\frac{1}{2}|4-x+ct|$

④ $x+ct<-4 \Rightarrow x-ct<-4$

$\Rightarrow f(x-ct)=0=f(x+ct)$

$\Rightarrow u(x,t)=0$

⑤ $x-ct>4 \Rightarrow x+ct>4$

$\Rightarrow f(x-ct)=0=f(x+ct)$

$\Rightarrow u(x,t)=0$

⑥ $x-ct<-4$ and $x+ct>4$

$\Rightarrow f(x-ct)=0=f(x+ct)$

$u(x,t)=0$

$$u(x,t)=\begin{cases}\frac{1}{2}(|4-x+ct|+|4-x-ct|), & -4+ct<x<4-ct\\ \frac{1}{2}|4-x-ct|, & -4-ct<x<4-ct \ \&\ x<-4+ct\\ \frac{1}{2}|4-x+ct|, & -4+ct<x<4+ct \ \&\ x>4-ct\\ 0, & \text{otherwise}\end{cases}$$

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