Homework #7 Solutions
Math 182, Spring 2009
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[A] 1. Solve $u_{tt} = c^2 u_{xx}, \quad x > 0$

$u(x,0) = 0, \; u_t(x,0) = 0$

$u(0,t) = h(t)$

$u(x,t) = F(x+ct) + G(x-ct)$

From the initial conditions, for $x > 0$:

$F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(s)\,ds, \; x > 0$

$G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds, \; x > 0$

where $f(x) = 0, \; g(x) = 0$

$\Rightarrow F(x) = 0, \; x > 0$

$G(x) = 0, \; x > 0$

Now, $u(x,t) = F(x+ct) + G(x-ct)$

$F(x+ct) = 0 \quad (\text{since } x+ct > 0)$

$G(x-ct) = ?$

$x - ct > 0 \Rightarrow F(x-ct) = 0$

$x - ct < 0 \Rightarrow F(x-ct) = ?$

Use the boundary condition

$u(0,t) = h(t) \Rightarrow h(t) = F(-ct) + G(ct)$

$= F(-ct) + 0$

$\Rightarrow F(-ct) = h(t)$

So let $z = -ct \Rightarrow F(z) = h(-z/c), \; z < 0$

$\Rightarrow F(x-ct) = h((ct-x)/c) = h(t - x/c), \; x - ct < 0$

$\Rightarrow u(x,t) = h(t - x/c), \; x - ct < 0$

So

$$\boxed{u(x,t) = \begin{cases} 0, & x > ct \\ h\left(t - \frac{x}{c}\right), & x < ct \end{cases}}$$

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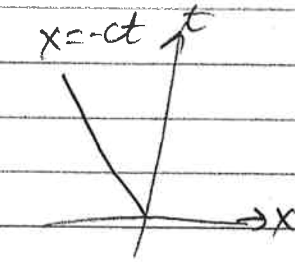
2. Solve $u_{tt} = c^2 u_{xx}$, $x<0$, $t>0$.

$u(x,0) = \cos x$, $x<0$

$u_t(x,0) = 0$, $x<0$

$u(0,t) = e^{-t}$, $t>0$.



Since $u(x,t) = F(x+ct) + G(x-ct)$

using the initial conditions we know that

$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds$, $x<0$

$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds$, $x<0$

So, since $x-ct<0$,

$G(x-ct) = \frac{1}{2}\cos(x-ct)$

$F(x+ct) = ?$

$x+ct<0 \Rightarrow F(x+ct) = \frac{1}{2}\cos(x+ct)$

$\Rightarrow u(x,t) = \frac{1}{2}\cos(x+ct) + \frac{1}{2}\cos(x-ct)$

$x+ct>0$? Use the boundary condition =

$u(0,t) = e^{-t} \Rightarrow e^{-t} = F(ct) + G(-ct)$

$F(ct) = e^{-t} - G(-ct)$

$F(z) = e^{-z/c} - G(-z)$, $z>0$

So if $x+ct>0$, $F(x+ct) = e^{-(x+ct)/c} - G(-x-ct)$

$= e^{-t-x/c} - \frac{1}{2}\cos(-x-ct)$

$= e^{-t-x/c} - \frac{1}{2}\cos(x+ct)$

$\Rightarrow u(x,t) = \frac{1}{2}\cos(x-ct) + e^{-t-x/c} - \frac{1}{2}\cos(x+ct)$

$= \frac{1}{2}(\cos(x-ct) + \cos(x+ct)) + e^{-t-x/c}$

So $\boxed{u(x,t) = \begin{cases} \frac{1}{2}(\cos(x-ct) + \cos(x+ct)), & x < -ct \\ \frac{1}{2}(\cos(x-ct) - \cos(x+ct)) + e^{-t-x/c}, & x > -ct \end{cases}}$

3. Solve $u_{tt} = c^2 u_{xx}$, $x > 0$, $t > 0$

$u(x,0) = 0$, $x > 0$

$u_t(x,0) = 0$, $x > 0$

$u_x(0,t) = h(t)$

Since $u(x,t) = F(x+ct) + G(x-ct)$

using the initial conditions we know that

$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds$, $x > 0$

$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds$, $x > 0$

where $f(x) = 0$, $g(x) = 0$

$\Rightarrow F(x) = 0$, $x > 0$

$G(x) = 0$, $x > 0$

$x + ct > 0 \Rightarrow F(x+ct) = 0$

$G(x-ct) = ?$

$x - ct > 0 \Rightarrow G(x-ct) = 0 \Rightarrow u(x,t) = 0$

$x - ct < 0 \rightarrow$ need to use boundary condition

$u_x(0,t) = h(t) \Rightarrow h(t) = \frac{1}{c}\frac{dF}{dt}(ct) - \frac{1}{c}\frac{dG}{dt}(-ct)$

$\Rightarrow h(t) = -\frac{1}{c}\frac{dG}{dt}$

$\Rightarrow \frac{dG}{dt}(-ct) = -ch(t)$

So for $z < 0$: $\frac{dG}{dz}(z) = h(-z/c)$

Find $G(x-ct)$: $\int_0^{x-ct} \frac{dG}{dz}dz = \int_0^{x-ct} h(-z/c)dz$

$G(x-ct) - G(0) = -c\int_0^{t-x/c} h(s)ds$

$G(0) = 0 \Rightarrow G(x-ct) = -c\int_0^{t-x/c} h(s)ds$

$\Rightarrow u(x,t) = -c\int_0^{t-x/c} h(s)ds$

So $\boxed{u(x,t) = \begin{cases} 0, & x > ct \\ -c\int_0^{t-x/c} h(s)ds, & x < ct \end{cases}}$

[B] Solve using the method of characteristics

$$u_{tt} = c^2 u_{xx}, \quad 0<x<L, \; t>0$$
$$u(x,0)=0, \; u_t(x,0)=0, \quad 0<x<L$$
$$u(0,t)=h(t), \; u(L,t)=0$$

$$u(x,t) = F(x+ct) + G(x-ct)$$

Using the initial conditions

$$F(x) = \tfrac{1}{2}f(x) + \tfrac{1}{2c}\int_0^x g(s)\,ds, \quad 0<x<L$$
$$G(x) = \tfrac{1}{2}f(x) - \tfrac{1}{2c}\int_0^x g(s)\,ds, \quad 0<x<L$$

where $f(x)=0, g(x)=0$

$$\Rightarrow F(x)=0, \quad 0<x<L$$
$$G(x)=0, \quad 0<x<L$$

Possibilities: $\underline{0<x+ct<L}$ | $\underline{0<x-ct<L}$

$F(x+ct)=0$ | $G(x-ct)=0$

$\underline{0<x-ct<L \text{ and } x+ct>L:}$

Need to use the boundary condition $u(L,t)=0$

$$u(L,t) = F(L+ct) + G(L-ct) = 0$$
$$F(L+ct) = -G(L-ct)$$
$$G(L-ct)=0 \Rightarrow F(L+ct)=0. \text{ Let } z=L+ct.$$
$$\Rightarrow F(z)=0, \; z>L \Rightarrow F(x+ct)=0$$

$\underline{x-ct<0, \; 0<x+ct<L}$

Need to use the boundary condition $u(0,t)=h(t)$

$$h(t) = F(ct) + G(-ct)$$
$$F(ct)=0 \Rightarrow G(-ct)=h(t). \text{ Let } z=-ct$$
$$\Rightarrow G(z) = h(-z/c), \quad z<0$$
$$\Rightarrow G(x-ct) = h(-(x-ct)/c) = h(t-x/c)$$

$\underline{x-ct<0, \; x+ct>L:}$ Use $F(x+ct)$ and $G(x-ct)$ found above.

$$\Rightarrow F(x+ct)=0 \text{ and } G(x-ct)=h(t-x/c)$$

Solution: $u(x,t) = F(x+ct) + G(x-ct)$

So, $$u(x,t) = \begin{cases} h\left(t-\frac{x}{c}\right), & x<ct \\ 0, & \text{otherwise} \end{cases}$$

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