Homework #8 Solutions
Math 182, Spring 2009
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$\boxed{A}$ 1. Solve $\begin{cases} u_{tt} - c^2 u_{xx} = \sin(kx-\omega t), & \omega/k \neq \text{const}, \ -\infty < x < \infty, \ t>0 \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases}$

$$\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin(ks-\omega\tau)\,ds\,d\tau \\
&= \frac{1}{2c}\int_0^t \left(-\frac{1}{k}\cos(ks-\omega\tau)\Big|_{x-c(t-\tau)}^{x+c(t-\tau)}\right)d\tau \\
&= \frac{1}{2c}\int_0^t \left(-\frac{1}{k}\cos(k(x+c(t-\tau))-\omega\tau)+\frac{1}{k}\cos(k(x-c(t-\tau))-\omega\tau)\right)d\tau \\
&= \frac{1}{2ck}\int_0^t \cos(k(x-ct)+(kc-\omega)\tau)-\cos(k(x+ct)-(kc+\omega)\tau)\,d\tau \\
&= \frac{1}{2kc}\left(\frac{1}{kc-\omega}\sin(k(x-ct)+(kc-\omega)\tau)\Big|_0^t \right. \\
&\qquad \left. + \frac{1}{kc+\omega}\sin(k(x+ct)-(kc+\omega)\tau)\Big|_0^t\right)
\end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{2kc}\left[\frac{1}{kc-\omega}\left(\sin(kx-\omega t)-\sin(k(x-ct))\right) + \frac{1}{kc+\omega}\left(\sin(kx-\omega t)-\sin(k(x+ct))\right)\right]}$$

2. p95: 4.9. Solve the problem

$$\begin{aligned}
&u_{tt}-u_{xx}=1, && -\infty<x<\infty,\ t>0 \\
&u(x,0)=x^2 && -\infty<x<\infty \\
&u_t(x,0)=1, && -\infty<x<\infty.
\end{aligned}$$

$u=u_1+u_2$:

u_1 solves
$$\begin{aligned}
&u_{1,tt}-u_{1,xx}=0 \\
&u_1(x,0)=x^2=f(x) \\
&u_{1,t}(x,0)=1=g(x)
\end{aligned}$$

$$\begin{aligned}
u_1(x,t) &= \frac{1}{2}(f(x-t)+f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\
&= \frac{1}{2}\left((x-t)^2+(x+t)^2\right)+\frac{1}{2}\int_{x-t}^{x+t} 1\,ds \\
&= \frac{1}{2}(x^2-2tx+t^2+x^2+2tx+t^2)+\frac{1}{2}(2t) \\
&= x^2+t^2+t
\end{aligned}$$

u_2 solves:
$$\begin{aligned}
&u_{2,tt}-u_{2,xx}=1 \\
&u_2(x,0)=0,\ u_{2,t}(x,0)=0
\end{aligned}$$

$$\begin{aligned}
u_2(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} 1\,ds\,d\tau \\
&= \frac{1}{2}\int_0^t \left(s\Big|_{x-(t-\tau)}^{x+(t-\tau)}\right)d\tau
\end{aligned}$$

$$u_2(x,t) = \int_0^t (t-\tau)\,d\tau$$
$$= t\tau - \tfrac{1}{2}\tau^2 \Big|_0^t$$
$$= \tfrac{1}{2}t^2$$

$$u(x,t) = u_1(x,t) + u_2(x,t)$$
$$\Rightarrow u(x,t) = x^2 + t^2 + t + \tfrac{1}{2}t^2$$
$$\boxed{u(x,t) = x^2 + \tfrac{3}{2}t^2 + t}$$

3. p. 96: 4.16. Solve the problem

$$u_{tt} - u_{xx} = xt, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = 0, \quad -\infty < x < \infty$$
$$u_t(x,0) = e^x, \quad -\infty < x < \infty$$

$u = u_1 + u_2$

u_1 solves: $u_{1,tt} - u_{1,xx} = 0$
$$u_1(x,0) = 0 = f(x)$$
$$u_{1,t}(x,0) = e^x = g(x)$$
$$u_1(x,t) = \tfrac{1}{2}(f(x-t) + f(x+t)) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$
$$= \tfrac{1}{2}\int_{x-t}^{x+t} e^s\,ds$$
$$= \tfrac{1}{2}(e^{x+t} - e^{x-t})$$

u_2 solves: $u_{2,tt} - u_{2,xx} = xt$
$$u_2(x,0) = 0$$
$$u_{2,t}(x,0) = 0$$
$$u_2(x,t) = \tfrac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} s\tau\,ds\,d\tau$$
$$= \tfrac{1}{4}\int_0^t \left(s^2\tau \Big|_{x-(t-\tau)}^{x+(t-\tau)}\right) d\tau$$
$$= \tfrac{1}{4}\int_0^t \tau\left[(x+(t-\tau))^2 - (x-(t-\tau))^2\right] d\tau$$
$$= x\int_0^t \tau t - \tau^2\,d\tau$$
$$= x\left(\tfrac{1}{2}t\tau^2 - \tfrac{1}{3}\tau^3\right)\Big|_0^t$$
$$= \tfrac{1}{6}xt^3$$

$$u(x,t) = u_1(x,t) + u_2(x,t)$$
$$\boxed{u(x,t) = \frac{e^{x+t} - e^{x-t}}{2} + \frac{xt^3}{6}}$$

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$\boxed{B}$ 1. Consider the PDE's below and determine the ODE's implied by the method of separation of variables

a) $u_t = K u_{xx} - V_0 u_x$

Let $u(x,t) = \varphi(x) T(t)$

$$\Rightarrow \frac{\partial u}{\partial t} = \varphi \frac{dT}{dt}$$

$$\frac{du}{\partial x} = \frac{d\varphi}{dx} \cdot T$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\varphi}{dx^2} \cdot T$$

$$\Rightarrow \varphi \frac{dT}{dt} = K \frac{d^2\varphi}{dx^2} T - V_0 \frac{d\varphi}{dx} T$$

Divide by φT:

$$\frac{1}{T}\frac{dT}{dt} = \frac{K}{\varphi}\frac{d^2\varphi}{dx^2} - \frac{V_0}{\varphi}\frac{d\varphi}{dx} = \text{const.} = -\lambda$$

$$\Rightarrow \frac{dT}{dt} = -T\lambda \text{ and } K\frac{d^2\varphi}{dx^2} - V_0\frac{d\varphi}{dx} = -\lambda\varphi$$

b) $u_{xx} + u_{yy} = 0$

Let $u(x,y) = \varphi(x)\psi(y)$

$$\Rightarrow \frac{d^2u}{\partial x^2} = \frac{d^2\varphi}{dx^2} \cdot \psi \text{ and } \frac{d^2u}{dy^2} = \varphi \cdot \frac{d^2\psi}{dy^2}$$

$$\Rightarrow \psi \frac{d^2\varphi}{dx^2} + \varphi \frac{d^2\psi}{dy^2} = 0$$

Divide by $\varphi\psi$:

$$\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} + \frac{1}{\psi}\frac{d^2\psi}{dy^2} = 0 \Rightarrow \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\frac{1}{\psi}\frac{d^2\psi}{dy^2} = \text{const.} = -\lambda$$

$$\Rightarrow \frac{d^2\varphi}{dx^2} = -\lambda\varphi \text{ and } \frac{d^2\psi}{dy^2} = \lambda\psi$$

c) $du/\partial t = \frac{K}{r^2} \partial/\partial r\left(r^2 \, du/\partial r\right)$

Let $u(r,t) = R(r) T(t)$

$$\Rightarrow \frac{du}{\partial t} = R \cdot \frac{dT}{dt} \text{ and } \frac{\partial}{\partial r}\left(r^2 \frac{du}{\partial r}\right) = \frac{d}{\partial r}\left(r^2 \cdot T \cdot \frac{dR}{dr}\right)$$

$$= T \cdot d/dr\left(r^2 \cdot dR/dr\right)$$

$$= T\left(r^2 R'\right)'$$

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So we have $R\frac{dT}{dt} = \frac{k}{r^2} T (r^2 R')'$

Divide by kRT: $\frac{1}{kT} \cdot \frac{dT}{dt} = \frac{1}{r^2 R} \cdot (r^2 R')' = \text{constant} = -\lambda$

$\Rightarrow \frac{dT}{dt} = -k\lambda T$ and $(r^2 R')' = -\lambda r^2 R$

$2rR' + r^2 R'' = -\lambda r^2 R$

$\Rightarrow rR'' + 2R' + \lambda r R = 0$

2. Consider the ODE $\phi'' + \lambda \phi = 0$. Determine the eigenvalues λ and corresponding eigenfunctions ϕ if ϕ satisfies:

a) $\phi'(0) = 0$, $\phi'(L) = 0$

$\underline{\lambda > 0}$: $\phi = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$\Rightarrow \phi' = \sqrt{\lambda}(-c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x))$

$\phi'(0) = 0 \Rightarrow 0 = \sqrt{\lambda} c_2 \Rightarrow c_2 = 0$

$\phi'(L) = 0 \Rightarrow -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0$

$\lambda \neq 0$ and $c_1 \neq 0 \Rightarrow \sin(\sqrt{\lambda} L) = 0$

$\Rightarrow \sqrt{\lambda} L = n\pi, \quad n = 1, 2, \ldots$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

$\phi_n(x) = \cos(\sqrt{\lambda} x) = \cos\left(\frac{n\pi}{L} x\right)$

$\underline{\lambda = 0}$: $\phi = c_1 x + c_2$

$\phi' = c_1$

$\phi'(0) = 0 \Rightarrow c_1 = 0 \Rightarrow \phi(x) = c_2$

So $\lambda_0 = 0 \Rightarrow \phi_0(x) = 1$

$\underline{\lambda < 0}$: $\phi = c_1 e^{-\sqrt{-\lambda} x} + c_2 e^{\sqrt{-\lambda} x}$

$\phi' = -\sqrt{-\lambda} c_1 e^{-\sqrt{-\lambda} x} + \sqrt{-\lambda} c_2 e^{\sqrt{-\lambda} x}$

$0 = \phi'(0) = -\sqrt{-\lambda} c_1 + \sqrt{-\lambda} c_2 \Rightarrow c_1 = c_2$

$0 = \phi'(L) = -\sqrt{-\lambda} c_1 e^{-\sqrt{-\lambda} L} + \sqrt{-\lambda} c_2 e^{\sqrt{-\lambda} L}$

$= -c_1 \sqrt{-\lambda} e^{-\sqrt{-\lambda} L} + c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda} L}$

$= c_1 \underbrace{\sqrt{-\lambda}}_{\neq 0} \underbrace{\left(e^{\sqrt{-\lambda} L} - e^{-\sqrt{-\lambda} L}\right)}_{\neq 0} = 0 \Rightarrow c_1 = 0$

$\Rightarrow \phi \equiv 0$.

Not an eigenfunction

So, eigenvalues and associated eigenfunctions:

$\lambda_0 = 0,\ \varphi_0 = 1$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2;\ \varphi_n = \cos\left(\frac{n\pi}{L}x\right),\ n = 1, 2, \ldots$

b) $\varphi'(0) = 0,\ \varphi(L) = 0$

$\underline{\lambda > 0}$: $\varphi = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\varphi' = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2\cos(\sqrt{\lambda}x)$

$\varphi'(0) = 0 \Rightarrow \sqrt{\lambda}c_2 = 0 \Rightarrow c_2 = 0$

$\varphi(L) = 0 \Rightarrow c_1\cos(\sqrt{\lambda}L) = 0$

$\Rightarrow \cos(\sqrt{\lambda}L) = 0$

$\sqrt{\lambda}L = \frac{(2n-1)\pi}{2},\ n = 1, 2, \ldots$

$\lambda_n = \left(\frac{(2n-1)}{2L}\pi\right)^2,\ n = 1, 2, \ldots$

$\varphi_n = \cos\left(\frac{(2n-1)\pi}{2L}x\right),\ n = 1, 2, \ldots$

$\underline{\lambda = 0}$: $\varphi = c_1x + c_2 \Rightarrow \varphi'(x) = c_1$

$\left.\begin{array}{l}\varphi'(0) = 0 \Rightarrow 0 = c_1 \\ \varphi(L) = 0 \Rightarrow c_2 = 0\end{array}\right\} \Rightarrow \varphi \equiv 0$ Not an eigenfunction

$\underline{\lambda < 0}$: $\varphi = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$

$\varphi' = -\sqrt{-\lambda}c_1e^{-\sqrt{-\lambda}x} + \sqrt{-\lambda}c_2e^{-\sqrt{-\lambda}x}$

$\varphi'(0) = 0 \Rightarrow -\sqrt{-\lambda}c_1 + \sqrt{-\lambda}c_2 = 0 \Rightarrow c_2 = c_1$

$\varphi(L) = 0 \Rightarrow c_1e^{-\sqrt{-\lambda}L} + c_2e^{\sqrt{-\lambda}L} = 0$

$c_1e^{-\sqrt{-\lambda}L} + c_1e^{\sqrt{-\lambda}L} = 0$

$c_1\underbrace{(e^{-\sqrt{-\lambda}L} + e^{\sqrt{-\lambda}L})}_{\neq 0} = 0 \Rightarrow c_1 = 0$

$\Rightarrow \varphi \equiv 0$

Not an eigenfunct

So: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2,\ n = 1, 2, 3, \ldots$

and $\varphi_n = \cos\left(\frac{(2n-1)\pi}{2L}x\right),\ n = 1, 2, \ldots$

c) $\varphi(0)=0,\ \varphi'(L)=0$

$\underline{\lambda>0}$: $\varphi=c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x)$

$\varphi'=-\sqrt{\lambda}c_1\sin(\sqrt{\lambda}x)+\sqrt{\lambda}c_2\cos(\sqrt{\lambda}x)$

$\varphi(0)=0 \Rightarrow c_1=0$

$\varphi'(L)=0 \Rightarrow \sqrt{\lambda}c_2\cos(\sqrt{\lambda}L)=0$

$\Rightarrow \cos(\sqrt{\lambda}L)=0$

$\Rightarrow \sqrt{\lambda}L=\frac{(2n-1)\pi}{2},\ n=1,2,\ldots$

$\Rightarrow \lambda_n=\left(\frac{(2n-1)\pi}{2}\right)^2,\ n=1,2,\ldots$

and $\varphi_n=\sin\left(\frac{(2n-1)}{2}\pi x\right)^2,\ n=1,2,\ldots$

$\underline{\lambda=0}$: $\varphi=c_1x+c_2$

$\varphi'=c_1$

$\varphi(0)=0 \Rightarrow c_2=0$

$\varphi'(L)=0 \Rightarrow c_1=0$

$\Rightarrow \varphi\equiv 0$ Not an eigenfunction

$\underline{\lambda<0}$: $\varphi=c_1e^{-\sqrt{-\lambda}x}+c_2e^{\sqrt{-\lambda}x}$

$\varphi'=-\sqrt{-\lambda}c_1e^{-\sqrt{-\lambda}x}+\sqrt{-\lambda}c_2e^{\sqrt{-\lambda}x}$

$\varphi(0)=0 \rightarrow 0=c_1+c_2 \Rightarrow c_2=-c_1$

$\varphi'(L)=0 \rightarrow 0=-\sqrt{-\lambda}c_1e^{-\sqrt{-\lambda}L}+\sqrt{-\lambda}c_2e^{\sqrt{-\lambda}L}$

$0=-\sqrt{-\lambda}c_1e^{-\sqrt{-\lambda}L}-\sqrt{-\lambda}c_1e^{\sqrt{-\lambda}L}$

$=-c_1\underbrace{\sqrt{-\lambda}}_{\neq 0}\underbrace{(e^{-\sqrt{-\lambda}L}+e^{\sqrt{-\lambda}L})}_{\neq 0}$

$\Rightarrow c_1=0 \Rightarrow \varphi\equiv 0$ Not an eigenfunction

So: $\lambda_n=\left(\frac{(2n-1)}{2L}\right)^2,\ n=1,2,\ldots$

and $\varphi_n=\sin\left(\frac{(2n-1)\pi}{2L}x\right),\ n=1,2,\ldots$

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