HW #9 Solutions
Math 182, Spring 2009
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A Solve the heat equation

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(0,t) = u(L,t) = 0 \end{cases} \quad (*)$$

with initial conditions:

1. $u(x,0) = 6\sin\left(\frac{9\pi}{L}x\right)$
2. $u(x,0) = 3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi}{L}x\right)$
3. $u(x,0) = \begin{cases} 1, & 0 < x < L/2 \\ 2, & L/2 < x < L \end{cases}$

From class, the formal solution of $(*)$ is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where B_n is to be determined by the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

1. $f(x) = 6\sin\left(\frac{9\pi}{L}x\right) \Rightarrow B_n = \frac{2}{L}\int_0^L 6\sin\left(\frac{9\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$

$$= \begin{cases} \frac{2}{L}\cdot 6 \cdot \frac{L}{2}, & n = 9 \\ 0, & n \neq 9 \end{cases}$$

$$\Rightarrow B_n = \begin{cases} 6, & n = 9 \\ 0, & n \neq 9 \end{cases}$$

So $\boxed{u(x,t) = 6\sin\left(\frac{9\pi}{L}x\right)e^{-k\left(\frac{9\pi}{L}\right)^2 t}}$

2. $f(x) = 3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi}{L}x\right)$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L \left(3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi}{L}x\right)\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{6}{L}\underbrace{\int_0^L \sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx}_{=\begin{cases} L/2, & n=1 \\ 0, & n\neq 1\end{cases}} - \frac{2}{L}\underbrace{\int_0^L \sin\left(\frac{3\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx}_{=\begin{cases} L/2, & n=3 \\ 0, & n\neq 3\end{cases}}$$

So, $B_n = \begin{cases} \frac{6}{L}\cdot\frac{L}{2} = 3, & n = 1 \\ -\frac{2}{L}\cdot\frac{L}{2} = -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$

2 (cont.) So $\boxed{u(x,t) = 3\sin(\frac{\pi}{L}x)e^{-k(\frac{\pi}{L})^2 t} - \sin(\frac{3\pi}{L}x)e^{-k(\frac{3\pi}{L})^2 t}}$

3. $f(x) = \begin{cases} 1, & 0<x<L/2 \\ 2, & L/2<x<L \end{cases}$

$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$

$= \frac{2}{L}\int_0^{L/2}\sin\left(\frac{n\pi}{L}x\right)dx + \frac{4}{L}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx$

$= \frac{2}{L}\left(\frac{-L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2}\right) + \frac{4}{L}\left(\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}\right)$

$= \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right) + \frac{4}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)$

$= \frac{2}{n\pi} + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\cos(n\pi)$

So $B_n = \begin{cases} -4/n\pi, & n=2,6,10,\dots \\ 0, & n=4,8,12,\dots \\ 6/n\pi, & n \text{ odd} \end{cases}$

So $u(x,t) = \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi}{L}x)e^{-k(n\pi/L)^2 t}$ with B_n as above.

$\boxed{B}$ 1. Solve $\begin{cases} u_t = ku_{xx} \\ u_x(x,0) = u_x(L,0) = 0 \end{cases}$

subject to

a) $u(x,0) = \begin{cases} 0, & 0<x<L/2 \\ 1, & L/2<x<L \end{cases}$

b) $u(x,0) = 6 + 4\cos(\frac{3\pi}{L}x)$

c) $u(x,0) = -3\cos(\frac{8\pi}{L}x)$

From class, the formal solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(\tfrac{n\pi}{L}x)\, e^{-k(\frac{n\pi}{L})^2 t}$$

a) $f(x) = \begin{cases} 0, & 0<x<L/2 \\ 1, & L/2<x<L \end{cases}$

From class, $A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\left[\int_0^{L/2} 0\,dx + \int_{L/2}^{L} 1\,dx\right]$

$= \frac{1}{L}\cdot\frac{L}{2} = \frac{1}{2}$

$$A_n = 2/L \int_0^L f(x)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= 2/L \int_{L/2}^{L} \cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= (2/L)(L/n\pi)\sin\left(\tfrac{n\pi}{L}x\right)\Big|_{L/2}^{L}$$
$$= -2/(n\pi)\sin\left(\tfrac{n\pi}{2}\right)$$
$$\Rightarrow A_n = \begin{cases} -2/(n\pi), & n=1,5,9,\ldots \\ 2/(n\pi), & n=3,7,11 \\ 0, & n \text{ even} \end{cases}$$

So $u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

with A_n defined as above

b) $f(x) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)$
$$A_0 = \tfrac{1}{L}\int_0^L \left(6 + 4\cos\left(\tfrac{3\pi}{L}x\right)\right)dx$$
$$= \tfrac{1}{L}\int_0^L 6\,dx + \tfrac{4}{L}\int_0^L \cos\left(\tfrac{3\pi}{L}x\right)dx$$
$$= 6 + \tfrac{4}{L}\cdot\tfrac{L}{3\pi}\sin\left(\tfrac{3\pi}{L}x\right)\Big|_0^L$$
$$= 6$$
$$A_n = 2/L \int_0^L \left(6 + 4\cos\left(\tfrac{3\pi}{L}x\right)\right)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= 12/L \int_0^L \cos\left(\tfrac{n\pi}{L}x\right)dx + 8/L \int_0^L \cos\left(\tfrac{3\pi}{L}x\right)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= 8/L \int_0^L \cos\left(\tfrac{3\pi}{L}x\right)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= \begin{cases} 8/L \cdot L/2, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

So $\boxed{u(x,t) = 6 + 4\cos\left(\frac{3\pi}{L}x\right)e^{-k(3\pi/L)^2 t}}$

c) $f(x) = -3\cos\left(\frac{8\pi}{L}x\right)$
$$A_0 = \tfrac{1}{L}\int_0^L -3\cos\left(\tfrac{8\pi}{L}x\right)dx = 0$$
$$A_n = 2/L \int_0^L -3\cos\left(\tfrac{8\pi}{L}x\right)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= \tfrac{-6}{L}\int_0^L \cos\left(\tfrac{8\pi}{L}x\right)\cos\left(\tfrac{n\pi}{L}x\right)dx$$
$$= \begin{cases} \tfrac{-6}{L}\cdot \tfrac{L}{2}, & n=8 \\ 0, & n\neq 8 \end{cases}$$
$$\Rightarrow \boxed{u(x,t) = -3\cos\left(\tfrac{8\pi}{L}x\right)e^{-k(8\pi/L)^2 t}}$$

2. Solve the eigenvalue problem

$$\begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = \varphi(2\pi) \\ \varphi'(0) = \varphi'(2\pi) \end{cases}$$

$\underline{\lambda<0}$: $\varphi = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$

$\varphi' = -\sqrt{-\lambda}c_1 e^{-\sqrt{-\lambda}x} + \sqrt{-\lambda}c_2 e^{\sqrt{-\lambda}x}$

$\varphi(0) = \varphi(2\pi) \Rightarrow c_1 + c_2 = c_1 e^{-\sqrt{-\lambda}2\pi} + c_2 e^{\sqrt{-\lambda}2\pi}$

$\Rightarrow c_1(1 - e^{-\sqrt{-\lambda}\cdot 2\pi}) + c_2(1 - e^{\sqrt{-\lambda}\cdot 2\pi}) = 0 \quad (1)$

$\varphi'(0) = \varphi'(2\pi) \Rightarrow -\sqrt{-\lambda}c_1 + \sqrt{-\lambda}c_2 = -\sqrt{-\lambda}c_1 e^{-2\pi\sqrt{-\lambda}} + \sqrt{-\lambda}c_2 e^{2\pi\sqrt{-\lambda}}$

$\Rightarrow c_1(1 - e^{-2\pi\sqrt{-\lambda}}) - c_2(1 - e^{2\pi\sqrt{-\lambda}}) = 0 \quad (2)$

$(1)+(2) \Rightarrow 2c_1(1 - e^{-2\pi\sqrt{-\lambda}}) = 0$

$\Rightarrow 2c_1 = 0$ or $1 - e^{\cancel{-2\pi\sqrt{-\lambda}}} = 0$

$\Rightarrow c_1 = 0 \Rightarrow c_2(1 + e^{2\pi\sqrt{-\lambda}}) = 0 \Rightarrow c_2 = 0$

So $\varphi \equiv 0$ not eigenfunction, so $\lambda < 0$ not eigenvalues.

$\underline{\lambda = 0}$: $\varphi = c_1 + c_2 x$

$\varphi' = c_2$

$\varphi(0) = \varphi(2\pi) \Rightarrow c_1 = c_1 + 2\pi c_2 \Rightarrow c_2 = 0$

$\varphi'(0) = \varphi'(2\pi) \Rightarrow c_2 = c_2$ so $0 = 0$ ✓ $\Rightarrow c_1$ arbitrary

So $\lambda_0 = 0$ and $\varphi_0(x) = 1$

$\underline{\lambda > 0}$: $\varphi = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\varphi' = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2\cos(\sqrt{\lambda}x)$

$\varphi(0) = \varphi(2\pi) \Rightarrow c_1 = c_1\cos(2\pi\sqrt{\lambda}) + c_2\sin(2\pi\sqrt{\lambda})$

$\Rightarrow c_1(1 - \cos(2\pi\sqrt{\lambda})) - c_2\sin(2\pi\sqrt{\lambda}) = 0 \quad (a)$

$\varphi'(0) = \varphi'(2\pi)$: $\sqrt{\lambda}c_2 = -\sqrt{\lambda}c_1\sin(2\pi\sqrt{\lambda}) + \sqrt{\lambda}c_2\cos(2\pi\sqrt{\lambda})$

$\Rightarrow c_1\sin(2\pi\sqrt{\lambda}) + c_2(1 - \cos(2\pi\sqrt{\lambda})) = 0 \quad (b)$

(a), (b) has nonzero solution only if

$$0 = \begin{vmatrix} 1-\cos(2\pi\sqrt{\lambda}) & -\sin(2\pi\sqrt{\lambda}) \\ \sin(2\pi\sqrt{\lambda}) & 1-\cos(2\pi\sqrt{\lambda}) \end{vmatrix} = 1 - 2\cos(2\pi\sqrt{\lambda}) + \cos^2(2\pi\sqrt{\lambda}) + \sin^2(2\pi\sqrt{\lambda})$$

$$= 2 - 2\cos(2\pi\sqrt{\lambda})$$

$\Rightarrow \cos(2\pi\sqrt{\lambda}) = 1$

$\Rightarrow 2\pi\sqrt{\lambda} = 2\pi n, \ n = 1, 2, \ldots$

$\lambda_n = n^2, \ n = 1, 2, \ldots$

$\varphi_n = \begin{cases} \sin(nx) \\ \cos(nx) \end{cases}$ or $\varphi_n = a_n\cos(nx) + b_n\sin(nx)$

3. p.125, 5.6a): Using separation of variables, find a formal solution

$$u_t - Ku_{xx} = 0, \quad 0 < x < 2\pi, \ t > 0$$
$$u(0,t) = u(2\pi,t), \ u_x(0,t) = u_x(2\pi,t) \quad t \geq 0$$
$$u(x,0) = f(x), \quad 0 \leq x \leq 2\pi$$

where f is a smooth periodic function.

Let $u(x,t) = \varphi(x)T(t)$

Then $u_t = \varphi(x)\,T_t(t)$, $u_{xx} = T(t)\,\varphi_{xx}(x)$

$$\Rightarrow \varphi(x)T_t(t) - KT(t)\varphi_{xx}(x) = 0$$

$$\frac{1}{KT}\frac{dT}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda = \text{constant}$$

$$\Rightarrow \frac{dT}{dt} = -\lambda KT \ ; \ \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \ \begin{matrix}\varphi(0) = \varphi(2\pi) \\ \varphi'(0) = \varphi'(2\pi)\end{matrix}$$

$$\frac{dT}{dt} = -\lambda KT \Rightarrow T = ce^{-\lambda Kt}$$

From previous problem, $\lambda_0 = 0 \rightarrow \varphi_0 = 1$

$$\lambda_n = n^2 \Rightarrow \varphi_n = \begin{cases}\cos(nx) \\ \sin(nx)\end{cases}$$

So, $u_0(x,t) = 1 \cdot e^{-0(Kt)} = 1$

$$u_n(x,t) = \begin{cases}\cos(nx)e^{-n^2Kt} \\ \sin(nx)e^{-n^2Kt}\end{cases}, \ n = 1, 2, \ldots$$

By generalized superposition

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}[A_n\cos(nx) + B_n\sin(nx)]e^{-n^2Kt}$$

Find A_0, A_n, B_n using $u(x,0) = f(x)$: $u(x,0) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx))$

A_0: $\int_0^{2\pi} f(x)dx = \int_0^{2\pi} A_0 dx + \sum_{n=1}^{\infty}\int_0^{2\pi}[A_n\cos(nx) + B_n\sin(nx)]dx$

$$= A_0(2\pi)$$

$$\underline{\underline{A_0 = \frac{1}{2\pi}\int_0^{L} f(x)dx}}$$

A_n: $\int_0^{2\pi} f(x)\cos(mx)dx = \int_0^{2\pi} A_0\cos(nx)dx$

$$+\int_0 [A_n\cos(nx) + B_n\sin(nx)]\cos(mx)dx$$

$$= A_n\int_0^{2\pi}\cos(nx)\cos(mx)dx + B_n\int_0^{2\pi}\sin(nx)\cos(mx)dx$$

$\int_0^{2\pi} f(x)\cos(mx)dx \quad \pi \cdot A_m$

$$\underline{A_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)dx}$$

B_n: $\int_0^{2\pi} f(x)\sin(mx)dx = \int_0^{2\pi} A_0 \sin(mx)dx$

$$+\int_0^{2\pi}(A_n\cos(nx) + B_n\sin(nx))\sin(mx)dx$$

$$= A_n\int_0^{2\pi}\cos(nx)\sin(mx)dx + B_n\int_0^{2\pi}\sin(nx)\sin(mx)dx$$

$$= B_m \cdot \pi$$

$$\Rightarrow B_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)dx$$

So

$$\boxed{\begin{aligned} u(x,t) &= A_0 + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx))e^{-kn^2t} \\ \text{where } A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\ A_n &= \frac{1}{\pi}\int_0^{\pi} f(x)\cos(nx)dx, \quad n=1,2,\ldots \\ B_n &= \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)dx, \quad n=1,2,\ldots \end{aligned}}$$