1 Introduction

We start with some definitions.

- A **sequence** is a function whose domain is the set of all nonnegative integers and whose range is a subset of the real numbers.
- A **dynamical system** is a relationship among the terms in a sequence.
- A **numerical solution** is a table of values that satisfy the dynamical system.
- A **recurrence relation** is an equation of the form
  \[ a(n + 1) = f(a(n), a(n - 1), \ldots, n), \]
  where \( f \) is a function (i.e., it is a difference equation).
- A **discrete dynamical system** is a sequence defined by a recurrence relation.

A first-order discrete dynamical system is a sequence \( a(n) \) such that
\[ a(n + 1) = f(a(n)). \]
More commonly, this is written
\[ a(n + 1) - a(n) = g(a(n)). \]
This relationship is a **first-order difference equation**.

The components of a dynamical system, \( A(n) \) are

1. an equation for sequences representing \( A(n) \),
2. a well-defined time period \( n \), and
3. at least one starting value (an **initial condition**).

**Example:** \( a(n + 1) = 0.5a(n) + 0.1, \ a(0) = 0.1 \). This can be solved analytically to obtain the general formula
\[ a(n) = (0.5)^n a(0) + \frac{1}{5} - \frac{1}{5}(0.5)^n. \]

Useful Maple commands in this chapter are:

- \texttt{rsolve} - solves a difference equation
- \texttt{seq} - used to obtain numerical values in the solution.

Note that many dynamical systems do not have closed-form solutions.
2 Modeling Discrete Change

The goal of discrete dynamical systems is to explain certain discrete behaviors or make long-term predictions. We will focus right now on discrete dynamical systems that have only constant coefficients. For example, a second-order discrete dynamical system with constant coefficients may be written

\[ a(n + 2) = b_1 a(n + 1) + b_0 a(n), \]

where \( b_0 \) and \( b_1 \) are arbitrary constants.

**Example: Drug Dosage** (Maple worksheet drugdosage.mw)

A doctor prescribes a patient take a pill containing 250 mg of a certain drug every 4 hours. Assume the drug is immediately ingested into the bloodstream once taken. Also, assume that every 4 hours, the patient’s body eliminates 30% of the drug in his/her bloodstream. Suppose the patient had 0 mg of the drug in his bloodstream prior to taking the first pill. How much of the drug will be in his bloodstream after 72 hours?

**Step 1: Identify the Problem**

Determine the relationship between the amount of drug in the bloodstream and time.

**Step 2: Make Simplifying Assumptions**

**Variables**

- \( a(n) \) - amount of drug in the bloodstream after period \( n \), where \( n = 0, 1, 2, \ldots \) 4-hour period (mg)
- \( n \) - number of 4-hour time periods

**Assumptions**

- The patient is of normal size and health.
- There are no other drugs being taken that will affect the prescribed drug.
- There are no internal or external factors that will affect the drug absorption rate.
- The patient always takes the prescribed dosage at the correct time.

**Step 3: Construct the Model**

So, we have

- \( a(n + 1) = \) amount of drug in the bloodstream in the future (mg)
- \( a(n) = \) amount of drug currently in the bloodstream (mg)

Define change as follows

\[ \text{Change} = \text{dose} - \text{loss in the system} \]

\[ \text{Change} = 250 - 0.3a(n) \]
So,

Future = Present − Change

\[ a(n + 1) = a(n) − 0.30a(n) + 250 \]

\[ \implies a(n + 1) = 0.70a(n) + 250 \]

**Step 4: Solve and Interpret the Model**

We can solve this difference equation analytically to obtain

\[ a(n) = \frac{2500}{3} − \frac{2500}{3}(0.7)^n. \]

Alternately, in Maple we can do: \( \text{rsolve}(a(n+1) = 0.7a(n)+100, a(0) = 0). \)

We want to find the value after 72 hours, so

\[ n = \frac{72}{4} = 18, \text{ and } a(18) \approx 831.98 \text{ mg}. \]

We can plot the results, as well (see the Maple worksheet), and find the limit as \( n \to \infty \):

\[ \lim_{n \to \infty} a(n) = \frac{2500}{3} \approx 833.33 \text{ mg}. \]

If this is a safe and effective dosage level, then the dosage schedule is acceptable.

### 3 Equilibrium Values and Long-term Behavior

**Definition** The number \( a_e \) is called an **equilibrium value** or **fixed point** for a discrete dynamical system if \( a(k) = a_e \) for all values of \( k \) when the initial value is set to \( a_e \); i.e., \( a(k) = a_e \) is a constant solution to the recurrence relation for the dynamical system. (So, \( a_e \) is an equilibrium value for \( a(n + 1) = f(a(n), a(n − 1), ..., n) \) if and only if \( a_e = f(a_e, a_e, ..., a_e) \).

**Notes**

1. Not all discrete dynamical systems have equilibrium values.
2. Many discrete dynamical systems have equilibrium values the system may never achieve.

**Example: Drug Dosage Revisited**

Recall: The model was governed by the difference equation \( a(n + 1) = 0.7a(n) + 250 \). To determine the equilibrium value(s), let \( a(n) = a(n + 1) = a_e \). Then,

\[ a_e = 0.7a_e + 250 \]

\[ \implies a_e = \frac{250}{0.3} = \frac{2500}{3} \]
3.1 Graphically Analyzing Equilibrium Values

Analyze models of the form \( a(n + 1) = ra(n) + b \), where \( r \) and \( b \) are constants. (Maple worksheet: graphicalequilib.mw).

**Example:** \( a(n + 1) = ra(n) + 50 \)

Equilibrium value: \( a_e = \frac{50}{1 - r} \). Therefore, there are three cases.

**Case 1:** \( r > 1 \) In this case, \( a_e < 0 \). Assuming a physical system, this can never be attained. Example: \( r = 2 \). See the Maple worksheet.

**Case 2:** \( 0 < r < 1 \) In this case, \( a_e > 0 \). Example: \( r = 0.25 \). See the Maple worksheet.

**Case 3:** \( r < 0 \) In this case, \( a_e > 0 \). Example: \( r = -1.01 \). See the Maple worksheet.

Suppose \( b = 0 \). Then, the difference equation is \( a(n + 1) = ra(n) \), and the only equilibrium value is \( a_e = 0 \).

3.2 Stability and Long-term Behavior

Given a discrete dynamical system, if the values of \( a(n) \) converge to \( a_e \) for an initial condition close to \( a_e \), then \( a_e \) is called a stable equilibrium value, or an attracting fixed point (i.e., if \( \lim_{n \to \infty} a(n) = a_e \)).

**Example:** (Maple worksheet stability.drug.mw)

We revise our drug dosage problem to have the dosage of medicine be 240 mg (for ease of analysis). In the worksheet, we try two different initial conditions, and analyze the solution. The result is an apparent convergence to the equilibrium value of 800. To verify, solve the difference equation and take the limit as \( n \to \infty \).

4 Modeling Nonlinear Discrete Dynamical Systems

A nonlinear discrete dynamical system cannot be written in the form

\[ a(n + 1) = b_0 a(n) + b_1 a(n - 1) + \ldots + b_n, \]

where \( b_i \) may be functions of \( n \).

**Example:** Growth of a Yeast Culture

This example is from the textbook. Maple worksheet: yeastpop.mw.

5 Systems of Discrete Dynamical Systems

**Example:** Astronaut Docking Procedure

Astronauts in training are required to practice a docking maneuver under manual control. As a part of this maneuver it is required to bring an orbiting spacecraft to rest relative to another orbiting craft. The hand controls provide for variable acceleration and deceleration, and there is a device on board the ship that measures the rate of closing between the two vehicles. The following strategy has been proposed for bringing the ship to rest. First, look at the closing velocity. If it is zero, we are done. Otherwise, remember the closing velocity and look at the acceleration control. Move the acceleration control so that it is opposite to the closing velocity (i.e., if closing velocity is positive we slow down, and vice versa) and proportional in magnitude (i.e., we brake twice as fast if we are closing twice as fast). After a time, look at the closing velocity again and repeat the procedure. Under what circumstances would this strategy be effective?

**Step 1: Identify the Problem**

Identify the relationship between the velocity of the spacecraft and time, and determine if $v(n) \to 0$.

**Step 2: Make Simplifying Assumptions**

**Variables**

- $t_n =$ time of $n^{th}$ velocity observation (s)
- $v_n =$ velocity at time $t_n$ (m/s)
- $c_n =$ time to make $n^{th}$ control adjustment (s)
- $a_n =$ acceleration after $n^{th}$ adjustment (m/s$^2$)
- $w_n =$ wait before the $(n+1)^{st}$ observation (s)

**Assumptions**

- The time of the $(n+1)^{st}$ observation is equal to the sum of the current time, the time to make the $n^{th}$ control adjustment, and the wait before the $(n+1)^{st}$ observation: $t_{n+1} = t_n + c_n + w_n$
- The velocity at time $t(n+1)$ is equal to sum of the velocity at time $t_n$ and the contributions from the acceleration at times $t_{n-1}$ and $t_n$; i.e.,
  
  $$v_{n+1} = v_n + a_{n-1}c_n + a_n w_n.$$ 
- The spacecraft follows the standard control law, $a_n = -kv_n$.
- The control adjustment time $c_n$ and the wait time $w_n$ are constant; so, $c_n = c$ and $w_n = w$.

Given these assumptions, we see that

$$v_{n+1} - v_n = (-kv_{n-1})c_n + (-kv_n)w_n.$$ 

Since $c_n = c$ and $w_n = w,$

$$t_{n+1} - t_n = c + w,$$

so the time need not be included explicitly as a variable.

**Step 3: Construct the Model**
Define $x_1(n) = v_n$ and $x_2(n) = v_{n-1}$. This makes sense because our equation for $v_{n+1}$ involves both $v_n$ and $v_{n-1}$. Then, the system of difference equations that models this system is
\begin{align*}
x_1(n+1) - x_1(n) &= -kwx_1(n) - kcx_2(n) \\
x_2(n+1) - x_2(n) &= x_1(n) - x_2(n)
\end{align*}

**Step 4: Solve and Interpret the Model**

First, find the equilibrium solution(s):
\begin{align*}
x_{1e} - x_{1e} &= -kwx_{1e} - kcx_{2e} \\
x_{2e} - x_{2e} &= x_{1e} - x_{2e}
\end{align*}

There is one solution to this system of equations, $x_{1e} = x_{2e} = 0$. Now, we must determine if this equilibrium value is stable. Given the information we currently possess, we cannot determine if this equilibrium value is stable. Suppose, however, we know that $c \ll w$. If $v_n$ and $v_{n-1}$ are not too different, then $v_{n+1} - v_n \approx -kvw_n$ should be a reasonably accurate approximation. Solving, we see that $v_n = v_0(1 - kw)^n$. Thus, if $0 < kw < 1$ or $1 < kw < 2$, then $v_n \to 0$ as $n \to \infty$. Therefore, if the control adjustments are not too violent (and, so, the time between control adjustments is not too long), this control strategy will work.

6 Modeling Nonlinear Systems of DDS

**Example: Discrete SIR Model of Epidemics**

This example is found in the textbook. The Maple worksheet is discrsir.mw.

**Example: Battle Example** (Maple worksheet: battle_numer.mw)

Two forces, which we will call red and blue, are engaged in battle. In this conventional battle, attrition is due to direct fire (infantry) and area fire (artillery). The attrition rate due to direct fire is assumed proportional to the number of enemy infantry. The attrition rate due to artillery depends on both the amount of enemy artillery and the density of friendly troops. Red has amassed five divisions to attack a blue force of two divisions. Blue has the advantage of defense and superior weapon effectiveness. How much more effective does blue have to be in order to prevail in battle?

**Step 1: Identify the Problem**

Determine conditions of equilibrium so that blue wins.

**Step 2: Make Simplifying Assumptions**

**Variables**

- $R(n)$ = number of red units at time $n$ (divisions)
- $B(n)$ = number of blue units at time $n$ (divisions)
- $D_R$ = red attrition rate due to direct fire (units/hr)
- $D_B$ = blue attrition rate due to direct fire (units/hr)
- $I_R$ = red attrition rate due to artillery (units/hr)
• $I_B$ = blue attrition rate due to artillery (units/hr)
• $a_1$, $a_2$ = coefficient for red, blue effectiveness of direct fire (per hour)
• $b_1$, $b_2$ = coefficient for red, blue effectiveness of artillery (per unit per hour)
• $n$ = number of hours

Assumptions

• $D_R = a_1B$
• $D_B = a_2R$
• $I_R = b_1RB$
• $I_B = b_2RB$
• $R \geq 0$ and $B \geq 0$
• $R(0) = 5$, $B(0) = 2$
• $a_1$, $a_2$, $b_1$, $b_2 > 0$
• $a_1 > a_2$, $b_1 > b_2$
• $R(n) - R(n - 1) = D_R + I_R$, $B(n) - B(n - 1) = D_B + I_B$

Step 3: Construct the Model

Using the above assumptions, we obtain the following system:

\[
R(n) = R(n - 1) - a_1B(n - 1) - b_1R(n - 1)B(n - 1)
\]
\[
B(n) = B(n - 1) - a_2R(n - 1) - b_2R(n - 1)B(n - 1)
\]

We need numerical values for $a_i$ and $b_i$. None are given, so we need to “guess.” Suppose a battle lasts 5 days and engagement occurs 12 hours/day. Then one force is depleted in 60 hours of battle. If a force were depleted by 5% for 60 hours, the fraction left is $(0.95)^{60} = 0.05$. So, assume $a_2 = 0.05$. Since area fire is less effective than direct fire, assume $b_2 = 0.005$ (small because multiplied by both $R$ and $B$). Blue is supposed to have greater weapons effectiveness than red, so $a_1 > a_2$ and $b_1 > b_2$. Assume $a_1 = \lambda a_2$ and $b_1 = \lambda b_2$ for some $\lambda > 1$. We wish to find the smallest $\lambda$ such that $R(n) \to 0$ before $B(n) \to 0$. So, the difference equations are

\[
R(n) = R(n - 1) - \lambda(0.05)B(n - 1) - \lambda(0.005)R(n - 1)B(n - 1)
\]
\[
B(n) = B(n - 1) - 0.05R(n - 1) - 0.005R(n - 1)B(n - 1)
\]