1 Proportionality

Recall: $y$ is proportional to $x$ ($y \propto x$) if $y = kx$. We will assume for the purposes of our discussions that $k > 0$. Geometrically, the proportionality relationship $y = kx$ represents a line of slope $k$ passing through the origin.

In addition, there is a transitive rule for proportionality:

$$y \propto x \text{ and } x \propto z \implies y \propto z.$$

1.1 Testing and Estimating Proportionality Arguments

The following procedure is used to test and estimate proportionality arguments.

1. Enter data observed for the dependent and independent variables.
2. Plot the raw data points to check for trends and to identify potential data outliers.
3. Perform the transformations that support a hypothesized proportionality.
   → If, for example, you suspect $y \propto x^2$, you would look at the set of points $(x_i^2, y_i)$ to see if they form a line.
4. Plot the transformed data to test the hypothesized proportionality.
5. Estimate the constant of proportionality.
   • Estimate the slope directly by choosing two data points.
   • Find the slope using least squares.

*Some of the material in this handout is taken directly from *A First Course in Mathematical Modeling* by Frank Giordano, et al.
1.2 Example:

Consider a mass-spring system on which an experiment was conducted to measure the stretch of the spring as a function of the mass (measured as weight) placed on the spring. The data is given in the following table.

<table>
<thead>
<tr>
<th>Mass</th>
<th>Elongation</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.000</td>
</tr>
<tr>
<td>100</td>
<td>1.875</td>
</tr>
<tr>
<td>150</td>
<td>2.750</td>
</tr>
<tr>
<td>200</td>
<td>3.250</td>
</tr>
<tr>
<td>250</td>
<td>4.375</td>
</tr>
<tr>
<td>300</td>
<td>4.875</td>
</tr>
<tr>
<td>350</td>
<td>5.675</td>
</tr>
<tr>
<td>400</td>
<td>6.500</td>
</tr>
<tr>
<td>450</td>
<td>7.250</td>
</tr>
<tr>
<td>500</td>
<td>8.000</td>
</tr>
<tr>
<td>550</td>
<td>8.750</td>
</tr>
</tbody>
</table>

By Hooke’s law, $F = kx \implies x \propto F$ and $F = ma \implies F \propto m$. Therefore, $x \propto m$. A plot of the data is approximately linear, with the line approximately passing through the origin. We will estimate the slope in two ways: estimate the slope by choosing two points and estimate the slope using a least-squares approximation.

- Estimate the slope by choosing two points: choose (200, 3.25) and (300, 4.875). Then,
  
  $$k = \frac{4.875 - 3.25}{300 - 200} = 0.01625.$$

- Use a least-squares approximation to find the slope: $k \approx 0.016219$ (very close to the estimate).

2 Geometric Similarity

Geometric similarity is a concept related to proportionality and can be used to simplify the mathematical modeling process.

**Definition.** Two objects are said to be geometrically similar if there is a one-to-one correspondence between points of the objects such that the ratio of distances between corresponding points is constant for all possible pairs of points.

We will look at two examples to illustrate the usefulness of geometric similarity.
Example 1: Raindrop from a Cloud

Suppose that we are interested in the terminal velocity of a raindrop from a cloud.

Step 1: Identify the Problem. In this case, the problem is more or less clear. We wish to find the terminal velocity of a raindrop from a cloud. We will narrow this down a bit to say that we seek the terminal velocity of a raindrop from a motionless cloud.

Step 2: Make Simplifying Assumptions. Variables:

- \( m \) = mass of raindrop
- \( v \) = velocity (speed) of raindrop
- \( v_t \) = terminal velocity of raindrop
- \( S \) = surface area of raindrop
- \( F_d \) = drag force from air resistance
- \( F_g \) = gravitational force

Assumptions: One simplifying assumption was made in the problem statement; namely, that the cloud is motionless. If we draw a free-body diagram, we will see that the only forces acting on the raindrop are gravity (i.e., from the weight of the raindrop), given by \( F_g \), and drag (from air resistance), given by \( F_d \). We will assume that the air resistance is proportional to the product of the surface area \( S \) of the raindrop and the square of its speed \( v \) (so \( F_d \propto Sv^2 \)). By Newton’s second law, we have

\[ F = F_d - F_g = ma. \]

Under terminal velocity, \( a = 0 \), so we Newton’s second law gives us \( F_g = F_d \). The gravitational force is proportional to the mass of the raindrop, so \( F_g \propto m \).

Next, assume that all raindrops are geometrically similar, which allows us to relate area and volume so that

\[ S \propto l^2 \text{ and } V \propto l^3 \]

for any characteristic dimension \( l \). Thus, \( l \propto S^{\frac{1}{2}} \) and \( l \propto V^{\frac{1}{3}} \), which implies that

\[ S \propto V^{\frac{2}{3}}. \]

Since weight and mass are proportional to volume, the transitive rule for proportionality gives us

\[ S \propto m^{\frac{2}{3}}. \]

Step 3: Construct the Model. From above, we obtained that \( F_d = F_g \), so \( m \propto m^{\frac{2}{3}} v_t^2 \).

Solving for the terminal velocity gives

\[ m^{\frac{1}{6}} \propto v_t. \]

So, the raindrop’s terminal velocity is proportional to the mass of the raindrop, raised to the one-sixth power.
Example 2: Body Weight

Question: How much should an individual weigh?

A rule of thumb given to people who want to run a marathon is 2 lb. of body weight per inch of height. Tables have been designed to suggest weights for different purposes: healthy weights (for doctors); upper weight allowances (organizations concerned about physical conditioning), etc.

In this example, we will examine how height and weight should vary. However, before doing this, we need to take into account the fact that body weight does not just depend on height. For example, bone density could be a factor. Is there a significant variation in bone density from person to person? Is the volume of the bone relatively constant? What about differences (if any) in the density of bone, muscle, and fat?

In this example, we will consider bone density as a constant (by accepting an upper limit) – a simplifying assumption – and discuss how to predict weight as a function of height, gender, age, and body density.

Step 1: Identify the Problem. For various heights, genders, and age groups, determine upper weight limits that represent maximum levels of acceptability based on physical appearance.

Step 2: Make Simplifying Assumptions. As one simplifying assumption, assume that some parts of the body are composed of an inner and outer core of different densities, and that the inner core is composed primarily of bones and muscle and that the outer core is primarily a fatty material, giving rise to the different densities. Next, we will construct submodels to determine how the weight of each core might vary with height.

First, assume that for adults, certain parts of the body, such as the head, have the same volume and density for different people. So, the weight of an adult is given by

\[ W = k_1 + W_{in} + W_{out}, \]  

(1)

where \( k_1 > 0 \) is the constant weight of those parts having the same volume and density for different individuals and \( W_{in} \) and \( W_{out} \) are the weights of the inner and outer core, respectively.

Look at the inner core. How do the volumes of the exremities and the trunk vary with height? People are not geometrically similar, since they do not appear to be scaled models of one another. By our definition of the problem, however, we are concerned with an upper weight limit based on appearance. Despite the seeming subjectiveness of this, it might seem reasonable to assume that what might be visualized as acceptable for a 6 ft. person might be a scaled image of a 5 ft. 5 in. person. So, for the purposes of our problem, geometric similarity of individuals is a reasonable assumption. Note that we are not assuming any particular shape; we are just assuming that the ratios of distances between corresponding points in individuals are the same. Under this assumption, the volume of each component we are considering is proportional to the cube of a characteristic dimension. We shall choose the height, \( h \), as the characteristic dimension. So, the sum of the components must be
proportional to the cube of the height, or
\[ V_{in} \propto h^3. \]  
(2)

Now, let us look at the average weight density of the inner core. Assuming that the inner core is composed of muscle and bone, each of which has a different density, what percentage of the total volume of the inner core is occupied by bones? If we assume that bone diameter is proportional to the height, then the total volume occupied by the bones would be proportional to the cube of the height. This implies that the percentage of the total volume of the inner core occupied by the bones in geometrically similar individuals is constant. What does this tell us about the average weight density \( \rho_{in} \)?

Consider the average weight density \( \rho_{avg} \) of a volume \( V \) consisting of two components \( V_1 \) and \( V_2 \), each having a density \( \rho_1 \) and \( \rho_2 \). Then, \( V = V_1 + V_2 \) and
\[ \rho_{avg} V = W = \rho_1 V_1 + \rho_2 V_2. \]
So,
\[ \rho_{avg} = \rho_1 \frac{V_1}{V} + \rho_2 \frac{V_2}{V}. \]
Thus, provided that \( \frac{V_1}{V} \) and \( \frac{V_2}{V} \) do not change, the average weight density is constant.

Application of the above analysis to the inner core gives
\[ W_{in} = V_{in} \rho_{in} \propto h^3, \]
or
\[ W_{in} = k_2 h^3 \text{ for } k_2 > 0. \]  
(3)

Now, look at the outer core. Since the table is to be based on appearance, it can be argued that the thickness of the outer core should be constant, regardless of the height. If \( \tau \) represents this thickness, then the weight of the outer core is
\[ W_{out} = \tau \rho_{out} S_{out}, \]
where \( S_{out} \) is the surface area of the outer core, and \( \rho_{out} \) is the density of the outer core. As before, assuming that the objects are geometrically similar, it follows that the surface area is proportional to the square of the height. If the density of the outer core of fatty material is assumed to be constant for all individuals, then we have
\[ W_{out} \propto h^2. \]
However, it may be argued that taller people can carry a greater thickness for the fatty layer. If it is assumed that the thickness of the outer core is proportional to the height, then
\[ W_{out} \propto h^3. \]
So, allowing both of these assumptions gives
\[ W_{out} = k_3 h^2 + k_4 h^3, \text{ where } k_3, k_4 \geq 0. \]  
(4)
Step 3: Construct the Model. Putting together the information from the submodels represented by Equations (1), (3), and (4) gives

\[ W = k_1 + k_3 h^2 + k_5 h^3 \text{ for } k_1, k_5 > 0 \text{ and } k_3 \geq 0, \]  

where \( k_5 = k_2 + k_4 \).

Step 4: Solve and Interpret the Model. This model invalidates the rules of a constant weight increase for each additional inch of height, the assumption that prompted this problem. In order to compare the two “models,” consider the following. Suppose we are looking at individuals in the age 17-21 category. Then, a 30-in. waist is judged the upper limit acceptable for the sake of personal appearance in a male with a height of 66 in. The 2-lb-per-inch rule would allow a 30-in. waist for a male with a height of 72 in., as well. On the other hand, the model based on geometric similarity suggests that all distances between corresponding points should increase by the same ratio. Thus, the male with a height of 72 in. should have a waist of

\[ 30 \left( \frac{72}{66} \right) = 32.7. \]

A comparison of the two models using the above analysis gives:

\[
\begin{array}{|c|c|c|}
\hline
\text{Height (in.)} & \text{Linear models (in., waist measure)} & \text{Geometric similarity model (in., waist measure)} \\
\hline
66 & 30 & 30.0 \\
72 & 30 & 32.7 \\
78 & 30 & 35.5 \\
84 & 30 & 38.2 \\
\hline
\end{array}
\]

Step 5: Verify the Model. We would need to test the submodels, and the model itself, in order to validate it. To do this, we would need to collect data on various statistics, such as body fat, waist size, and height.

2.1 Testing Geometric Similarity

Since the principle of geometric similarity requires that the ratio of distances between corresponding pairs of points be the same for all pairs of points, we can test that requirement to see if the objects in a given collection are geometrically similar.

For example, we know that circles are geometrically similar, but let us check this. If \( c \) denotes the circumference of a circle, \( d \) its diameter, and \( s \) the length of the arc along the circle subtended by a given (fixed) angle \( \theta \), then we know from geometry that

\[ c = \pi d \text{ and } s = \left( \frac{d}{2} \right) \theta. \]
Thus, for any two circles,
\[
\frac{c_1}{c_2} = \frac{\pi d_1}{\pi d_2} = \frac{d_1}{d_2}
\]
and
\[
\frac{s_1}{s_2} = \frac{(d_1/2)\theta}{(d_2/2)\theta} = \frac{d_1}{d_2}
\]

Thus, the ratio of distances between corresponding points as we go around any two circles is always the ratio of their diameters. This observation supports the reasonableness of the geometric similarity argument for the circles.