Basic Matrix Concepts and Operations – Section 3.4

Math 81, Applied Analysis
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1 Basic Matrix Concepts

Definition: A **matrix** is an array of numbers.

Example: Coeficient matrix of a system of equations. Given the system of equations

$$\begin{array}{rcrcrcr} 3x & - & 2y & + & z & = & 0 \\ 2x & + & 4y & & & = & -2 \end{array}$$

the coefficient matrix is

Notation:

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A is an $m \times n$ matrix, where m is the number of rows and n is the number of columns. If $m = n$, A is called a **square matrix**.

Vectors: Vectors come in two flavors,

- **(column) vector**

- **row vector**

Special matrices:

- **upper triangular** – a square matrix having all zeros below the diagonal. Any of the other entries may be zero. The basic form is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

  Note: The diagonal is the set of entries a_{ii}.

  Example:

- **lower triangular** – a square matrix having all zeros above the diagonal. Any of the other entries may be zero. The basic form is

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

  Example:

- **diagonal** – a square matrix having all zeros above and below the diagonal. Any of the entries on the diagonal may be zero. The basic form is

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

  Examples:

  - the identity matrix I (the diagonal matrix having all ones on the diagonal;
  - the zero matrix (the matrix having only zeros).

Examples:

2 Matrix Operations

2.1 Transposition

Definition: The **transpose** of a matrix A, denoted A^T, is defined by switching the rows and columns. In mathematical notation, it is defined as $a_{ij} \leftrightarrow a_{ji}$.

Examples:

Definitions:

- **symmetric** – A matrix A is symmetric if $A^T = A$.

  Example:

- **skew-symmetric** – A matrix A is skew-symmetric if $A^T = -A$.

  Example:

2.2 Equality of Matrices

$A = (a_{ij})$ and $B = (b_{ij})$ are equal if and only if

i) A and B have the same size, and

ii) $a_{ij} = b_{ij}$ for all $i,\ j$.

2.3 Matrix Addition/Subtraction

- Matrix addition and subtraction are defined only for matrices of the same size.
- The sum (difference) is found by adding (subtracting) the corresponding entries.

Example:

2.4 Scalar Multiplication

Let c be a scalar and A an $m \times n$ matrix. Then cA is found by multiplying each entry in A by c.

Example:

Some laws: Here, 0 denotes the **zero matrix** (i.e., the matrix whose entries are all zero). A and B are matrices and c, d, and k are scalars.

a) $A + B = B + A$

b) $A + 0 = A$

c) $A + (-A) = 0$

d) $c(A + B) = cA + cB$, $(c + d)A = cA + dA$

e) $(ck)A = c(kA)$

Transposition laws

i) $(A + B)^T = A^T + B^T$

ii) $(cA)^T = cA^T$

2.5 Matrix Multiplication

Definition: The product $C = AB$ of an $m \times n$ matrix A and an $r \times p$ matrix B is defined if and only if $r = n$, i.e., the number of rows of B must equal the number of columns of A. C is then the $m \times p$ matrix with entries

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

Idea:

Examples:

1) $A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

2) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & -2 & -3 \end{pmatrix}$

WARNING: Note that

1) $AB \neq BA$ in general.

2) $AB = 0$ does **not** necessarily imply

   (a) $A = 0$

   (b) $B = 0$

   (c) $BA = 0$

3) $AC = AD$ does **not** necessarily imply $C = D$.

Other properties of matrix products:

In the following, A, B, and C are matrices and k is a scalar.

i) $(kA)B = k(AB)$

ii) $A(BC) = (AB)C$

iii) $(A + B)C = AC + BC$

iv) $C(A + B) = CA + CB$

v) $(AB)^T = B^T A^T$