Basis and Dimension for Vector Spaces – Sections 4.4, 4.7

Math 81, Applied Analysis
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1 Basis and Dimension

Idea: A basis represents the "building blocks" of a vector space.

Example: $\hat{\mathbf{i}} = (1, 0)$ and $\hat{\mathbf{j}} = (0, 1)$ form a basis of $\mathbb{R}^2$.

Definition:

Example: $\mathbb{R}^2$

Is $\{(1,1),(0,1)\}$ the only basis of $\mathbb{R}^2$?

Theorem 1.

Example: $\mathbf{v}_1 = (1,2),\ \mathbf{v}_2 = (1,0),\ \mathbf{v}_3 = (0,1)$ are linearly dependent.
$((1,\ 2) = 1(1,\ 0) + 2(0,\ 1))$.

Theorem 2.

Definitions:

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Examples:

(1) $\dim(\mathbb{R}^2) = ?$.

(2) $\dim(M_{22}) = ?$.

(3) $\dim(P_n) = ?$.

Note: $\dim(\{\mathbf{0}\}) = 0$.

Idea so far:

Why do we care?

Theorem 3. *Let V be an n-dimensional vector space and let S be a subset of V. Then*

(a) If S is linearly independent and consists of n vectors, S is a basis for V.

(b) If S spans V and consists of n vectors, then S is a basis for V.

(c) If S is linearly independent, then S is contained in a basis for V.

(d) If S spans V, then S contains a basis for V.

Idea: If we know the dimension of a vector space, then determining if a set of vectors is a basis becomes easier:

Examples:

(1) Is $\{(1,0),(2,1),(-1,2)\}$ a basis of $\mathbb{R}^2$?

(2) Is $\{(1,0,1),(0,1,0),(1,-1,1)\}$ a basis of $\mathbb{R}^3$?

(3) Is $\{(1,1,1),(0,1,0)\}$ a basis for $\mathbb{R}^3$?

(4) Is $\{1-x,\ x^2,\ 1+2x\}$ a basis of P_2?

Example: Find a basis of the subspace S of M_{33} defined by the set of 3×3 symmetric matrices.

2 Basis for Solution Spaces

The procedure for determining the basis of a solution space will be illustrated by example.

Examples:

(1) Find a basis for the solution space of

$$3x_1 + 4x_2 + 5x_3 = 0$$
$$2x_1 + 3x_2 + 4x_3 = 0$$

(2) Find a basis for the solution space of

$$2x + 3y + z = 0.$$