Convolution and Discintuous Functions – Sections 10.4 and 10.5

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1 Convolution and Products of Transforms – Section 10.4

Definition: The **convolution** $f * g$ of the piecewise continuous functions f and g is defined for $t \geq 0$ as

Theorem 1 (The Convolution Theorem). *If $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and $|f(t)|, |g(t)| \leq Me^{ct}$, then the Laplace transform of $f * g$ exists for all $s > c$ and*

Example: Use convolutions to find the inverse Laplace transform of

$$G(s) = \frac{s}{s^4 + 2s^2 + 1}$$

Application to Differential Equations

Example: Solve the initial value problem

$$y'' + 3y' + 2y = e^{-t},\ y(0) = 0,\ y'(0) = 0$$

using convolution.

2 Piecewise Continuous Input Functions – Section 10.5

Definition: The **unit step function** is defined as follows

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Graphically,

Example: Find $u(t-1) - u(t-2)$.

Example: $f(t) = \sin t \left(u\left(t - \frac{\pi}{2}\right) - u(t-\pi)\right)$

Laplace Transform of the Unit Step Function

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

2.1 Translation on the t-axis

Theorem 2. *If* $\mathcal{L}\{f(t)\}$ *exists for* $s > c$ *then*

Proof.

□

Examples: Find $\mathcal{L}\{f(t)\}$ where

(1) $f(t) = \begin{cases} 0, & t < 1 \\ t, & t \geq 1 \end{cases}$

(2) $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \text{otherwise} \end{cases}$

Example (Application) A 1 kg mass is attached to a spring with spring constant 16 N/m suspended in a viscous fluid providing resistance of 8 kg/s. The system is initially at rest and a forcing function, $F(t) = 2N$, is applied until time $t = 2$ s. Find $y(t)$.