

Laplace Transform and Inverse Transform – Section 10.1

Math 81, Applied Analysis
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1 Laplace Transform and Inverse Transform – Section 10.1

Definition:

Examples:

(1) $f(t) = e^{at}$

$$(2) f(t) = t$$

1.1 Properties of Laplace Transform – Linearity

Theorem 1 (Linearity). *For any functions $f(t)$ and $g(t)$ having a Laplace transform and constants a and b ,*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof. (by definition)

□

Example: Determine the Laplace transform of

(1) $f(t) = 4e^{3t} + 2\sin(5t)$

(2) $\cos(\omega t), \sin(\omega t)$

1.2 Existence and Uniqueness

Theorem 2 (Existence). *If $f(t)$ is piecewise continuous for all $t \geq 0$ and satisfies*

$$|f(t)| \leq Me^{ct} \text{ for } t \geq T$$

for some constants c and M (i.e., $f(t)$ is of exponential order), then the Laplace transform of $f(t)$ exists for all $s > c$.

Corollary 1. *If $f(t)$ satisfies the hypotheses above, $\lim_{s \rightarrow \infty} F(s) = 0$.*

Theorem 3 (Uniqueness). *If the Laplace transform of a function exists, it is unique.*

1.3 Inverse Transforms

If $f(t)$ satisfies certain conditions (not to be discussed here), then

$$f(t) = \mathcal{L}^{-1} \{F(s)\},$$

where $\mathcal{L}^{-1} \{F(s)\}$ is the **inverse Laplace transform**.

Theorem 4 (Uniqueness of Inverse Laplace Transform). *Suppose $f(t)$ and $g(t)$ satisfy the conditions for existence of Laplace transform. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ where $f(t)$ and $g(t)$ are continuous.*

Notes:

The following page contains a table of Laplace transforms that you are responsible for memorizing:

Table 1: Table of Common Laplace Transforms

$f(t)$	Laplace Transform, $\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}, s > a$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(at)$	$\frac{s}{s^2 - a^2}$
$\sinh(at)$	$\frac{a}{s^2 - a^2}$

Examples:

$$(1) F(s) = \frac{3}{s^2+9}$$

$$(2) F(s) = \frac{6}{s^4}$$

$$(3) F(s) = \frac{s-4}{s^2-4}$$

$$(4) F(s) = \frac{1}{s^5}$$