Introduction to Systems of Linear Differential Equations – Chapter 7

Math 81, Applied Analysis
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1 First Order Systems of Differential Equations – Section 7.1

We can write a higher order linear differential equation as a system of first order equations.

Example: $x'' + p(t)x' + q(t)x = f(t)$

Our goal in this chapter will be to solve systems of first order constant coefficient differential equations using eigenvalue-eigenvector theory. First, we must write the systems of differential equations in matrix form.

2 Matrices and Systems of Linear Differential Equations – Section 7.2

We can write a system of first order differential equations in matrix form.

Examples:

(1)

$$\begin{aligned} x_1' &= 2x_1 - 3x_2 \\ x_2' &= 4x_1 + 2x_2 \end{aligned} \tag{1}$$

(2)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -q(t)x_1 - p(t)x_2 + f(t) \end{aligned}$$

Consider the homogeneous differential equation

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x}(t) \tag{2}$$

Theorem 1 (Superposition Principle)**.** *Given $\mathbf{x_1}(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ n solutions to (2) on the open interval I, if c_1, $c_2, \ldots, c_n$ are constants, then*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

is also a solution of (2) on I.

Proof.

$\square$

We can check the linear independence of solutions using the Wronskain:

If $\mathbf{x_1}, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are solutions of (2), their Wronskian is

$$W(t) = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix}$$

(i.e., write each solution $\mathbf{x}_i$ as a column vector).

Theorem 2. *Suppose $\mathbf{x_1}, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are solutions of the $n \times n$ system (2) on an open interval I. Suppose $P(t)$ is continuous on I. Let $W = W(t)$. Then $\mathbf{x_1}, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent if and only if $W \neq 0$ for all $t \in I$. They are linearly dependent if $W \equiv 0$ on I.*

If $\mathbf{x_1}, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n linearly independent solutions of the $n \times n$ system (2), then the general solution is

Initial Value Problems – Example: Given the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

a) Verify that $\mathbf{x}_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix}$ are solutions.

b) Show that $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent.

c) Solve the initial value problem.

Solution: