# Transformation of Initial Value Problems – Section 10.2

Math 81, Applied Analysis
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Goal: Use the Laplace transform to solve differential equations and systems of differential equations.

**Theorem 1.** *Suppose $f(t)$ is continuous for all $t \geq 0$, satisfies $|f(t)| \leq Me^{ct}$ for some constants $c$ and $M$, and has a derivative $f'(t)$ that is piecewise continuous for $t \geq 0$. Then, the Laplace transform of $f'(t)$ exists for $s > c$ and*

*Proof.*

□

We can apply this to successive derivatives.

In general:

**Corollary 1.** *Suppose $f(t)$ and its derivatives $f'(t), \ldots, f^{(n-1)}(t)$ are continuous for $t \geq 0$ and are of exponential order (i.e., $f^{(j)}(t) \leq Me^{ct}$ for some constants $c$ and $M$), and suppose $f^{(n)}(t)$ is piecewise continuous for $t \geq 0$. Then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists for $s > c$ and*

$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} &= s^n \mathcal{L}\left\{f(t)\right\} - s^{n-1} f(0) - \cdots - f^{(n-1)}(0) \\ &= s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)\end{aligned}$$

**Examples:**

(1) Find $\mathcal{L}\{t\}$ given $\mathcal{L}\{1\} = \dfrac{1}{s}$ .

$$f(t) = t \implies f'(t) = 1$$

(2) Find $\mathcal{L}\{t\cosh(at)\}$

We can use this idea to solve initial value problems.

**Examples:**

(1) Solve the initial value problem

$$y' + 3y = 10\sin t,\ \ y(0) = 0$$

using the Laplace transform.

(2) Solve the initial value problem

$$x'' + 4x' + 3x = -9t,\ \ x(0) = 0,\ x'(0) = 1$$

using the Laplace transform.

**Systems of Differential Equations**

We can also use the Laplace transform to solve systems of differential equations.

**Example:** Solve the initial value problem

$$\begin{array}{rcrcrl} x' & = & -x & + & y, & x(0) = 0 \\ y' & = & x & - & y, & y(0) = 1 \end{array}$$

**Transforms of Integrals**

**Theorem 2.** *If $f(t)$ is piecewise continuous on $t \geq 0$ and satisfies $|f(t)| \leq Me^{ct}$ for some constants $c$ and $M$, then*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}F(s) \text{ for } s > c$$

*Equivalently,*

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)d\tau.$$

*Proof.* (Idea of Proof). Verify that $g(t) = \int_0^t f(\tau)d\tau$ satisfies the conditions for Laplace transform. Then,

$$\begin{aligned} g'(t) &= f(t) \\ \implies \mathcal{L}\{g'(t)\} &= \mathcal{L}\{f(t)\} \\ \implies s\mathcal{L}\{g(t)\} - g(0) &= \mathcal{L}\{f(t)\} \\ \implies \mathcal{L}\{g(t)\} &= \frac{1}{s}\mathcal{L}\{f(t)\} \end{aligned}$$

□

**Example:** Find the function whose Laplace transform is

$$\frac{1}{s^2 + 4s}.$$