Nonhomogeneous Linear Differential Equations – Section 5.5

Math 81, Applied Analysis
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1 The Method of Undetermined Coefficients

We are looking at

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{1}$$

A general solution of (1) has the form (by Theorem 5 in Section 5.2 of the text)

$$y = y_c + y_p,$$

where
y_c:

y_p

Description of the Method of Undetermined Coefficients

Note:

Table 1: Table of Particular Solution "Guesses"

$f(x)$	Choice for y_p
$a_n x^n + \cdots + a_1 x + a_0$	$A_n x^n + \cdots + A_1 x + A_0$
ke^{ax}	Ae^{ax}
$k\cos(\omega x)$ or $k\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$(a_n x^n + \cdots + a_1 x + a_0)e^{ax}$	$(A_n x^n + \cdots + A_1 x + A_0)e^{ax}$
$(a_n x^n + \cdots + a_1 x + a_0)\cos(\omega x)$ or $(a_n x^n + \cdots + a_1 x + a_0)\sin(\omega x)$	$(A_n x^n + \cdots + A_1 x + A_0)\cos(\omega x) +$ $(B_n x^n + \cdots + B_1 x + B_0)\sin(\omega x)$
$ke^{ax}\cos(\omega x)$ or $ke^{ax}\sin(\omega x)$	$e^{ax}(A\cos(\omega x) + B\sin(\omega x))$
$e^{ax}(a_n x^n + \cdots + a_1 x + a_0)\cos(\omega x)$ or $e^{ax}(a_n x^n + \cdots + a_1 x + a_0)\sin(\omega x)$	$e^{ax}((A_n x^n + \cdots + A_1 x + A_0)\cos(\omega x) +$ $(B_n x^n + \cdots + B_1 x + B_0)\sin(\omega x))$

Examples: Find the general solution

(1) $y'' - y' - 2y = 8e^{3x}$

(2) $y'' - y' - 2y = 4x^2$

(3) $y'' - y' = 4x^2$

(4) $y'' - y' - 2y = -20\sin(2x) + 4e^x$

Example: Solve the initial value problem

$$y'' + 2y' + y = 2e^{-x},\ y(0) = 1, y'(0) = 0$$

2 Solution by Variation of Parameters

The method of undetermined coefficients, although having important engineering applications, is **only** valid for constant coefficient differential equations with special right hand side functions.

The method of variation of parameters is general, applying to all linear differential equations written in the form where the highest order derivative has a coefficient of 1. We will limit our discussion to second order linear differential equations. Thus, for our purposes, the method of variation of parameters will apply to all second order differential equations of the form

$$y'' + p(x)y' + q(x)y = f(x), \tag{2}$$

where $p(x),\ q(x),$ and $f(x)$ are arbitrary functions continuous on some interval (a, b).

Example: Find the general solution of

$$y'' - 2y' + y = e^x \ln x,\ \ x > 0$$