Homogeneous Higher Order Differential Equations – Sections 5.1-5.3

Math 81, Applied Analysis
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1 Second Order Linear Differential Equations: Introduction – Section 5.1

A general **second order linear differential equation** is a second order differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = F(x), \tag{1}$$

where $A(x)$, $B(x)$, $C(x)$, and $F(x)$ are arbitrary functions that are continuous on some open interval I. The requirement of continuity ensures that (1) will have a solution.

If $A(x) \neq 0$ on I, we can rewrite (1) as

$$y'' + p(x)y' + q(x)y = f(x). \tag{2}$$

The associated homogeneous differential equation is

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

Theorem 1 (The Superposition Principle) *If $y_1(x)$ and $y_2(x)$ solve (3), then*

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

also solves (3).

Proof: Plug $y(x) = c_1y_1(x) + c_2y_2(x)$ into (3).

1.1 General Solutions of Homogeneous Equations

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of (3), where $p(x)$ and $q(x)$ are continuous on some open interval I. Then, $y_1(x)$ and $y_2(x)$ form a basis of the solution space of (3) on I; i.e.,

is a **general solution** of (3) on I.

Definition: Given $f(x)$ and $g(x)$ continuous functions on an open interval I with continuous derivatives on I, the **Wronskian** is defined as

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Theorem 2 *Suppose $y_1(x)$ and $y_2(x)$ are solutions to (3) on an open interval I where $p(x)$ and $q(x)$ are continuous. Then $y_1(x)$ and $y_2(x)$ are linearly independent if and only if $W(y_1, y_2) \neq 0$ for all $x \in I$. Alternately, they are linearly dependent if and only if $W(y_1, y_2) \equiv 0$.*

Example: $y_1(x) = e^x,\ y_2(x) = e^{-x}$ solve $y'' - y = 0$. Are y_1 and y_2 linearly independent?

1.2 Existence and Uniqueness

Suppose that $p(x),\ q(x),$ and $f(x)$ are continuous on the open interval I. Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x),\ y(a) = y_0, y'(a) = \tilde{y}_0$$

has a unuique solution on I.

2 General Solutions of Higher Order Linear Equations – Section 5.2

Given the homogeneous linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0. \tag{4}$$

(i) If $y_1(x),\ y_2(x), \ldots, y_k(x)$ solve (4), then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

solves (4).

(ii) If $y_1(x),\ y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of (4) with $p_1(x),\ p_2(x), \ldots, p_n(x)$ continuous on some open interval I, then $\{y_1(x),\ y_2(x), \ldots, y_n(x)\}$ forms a basis for the solution space; i.e.,

is a general solution of (4) on I.

Note: The dimension of the solution space is the order of the differential equation.

(iii) Existence and uniqueness generalizes to n^{th} order linear initial value problems.

The definition of the Wronskian can be extended to n functions:

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

and can be used to check the linear independence/dependence of a set of solutions to (4), as in Theorem 2.

Example: Show that $y_1 = e^x,\ y_2 = \cos(x),\ y_3 = \sin(x)$ form a basis of the solution space of $y''' - y'' + y' - y = 0$.

3 Second Order Constant Coefficient Homogeneous Differential Equations – Section 5.3

Look at equations of the form

$$ay'' + by' + cy = 0, \tag{5}$$

where $a, \ b, \ c$ are constants.

There are three possibilities for the roots:

Case 1: $b^2 - 4ac > 0$ (two distinct real roots)
Case 2: $b^2 - 4ac = 0$ (one repeated real root, $r = -\frac{b}{2a}$)
Case 3: $b^2 - 4ac < 0$ (complex conjugate roots)

Case 1: Two Distinct Real Roots, r_1 and r_2

Example: Find the general solution of

$$y'' - 5y' + 6y = 0.$$

Case 2: One Real Double Root, $r = r_1 = r_2 = -\frac{b}{2a}$

Example: Solve the initial value problem

$$y'' - 4y' + 4y = 0, \; y(0) = 0, y'(0) = 2.$$

Case 3: Complex Conjugate Roots

Preliminary Stuff

Euler's formula:

$$e^{ix} = \cos x + i \sin x \tag{6}$$
$$e^{-ix} = \cos x - i \sin x \tag{7}$$

$\cos x$ and $\sin x$ can be obtained from (6) and (7):

Let $z = s + it$.

Examples:

1. $z = 2 \to \overline{z} =$
2. $z = 1 - 4i \to \overline{z} =$
3. $z = 2i \to \overline{z} =$

We can determine the magnitude of z by computing:

In addition, we can compute e^z as follows:

In our problem, we have

$$
\begin{aligned}
r_{1,2} &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \;\; b^2 - 4ac < 0 \\
\implies r_{1,2} &= -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a} \\
&= \alpha \pm i\beta
\end{aligned}
$$

Therefore, two solutions are

General solution:

Example: Find a general solution of

$$y'' + y' + y = 0$$